Proceedings of the 32nd Conference
on Decision and Control
San Antonio, Texas • December 1993

Steering left-invariant control systems on matrix Lie groups*

Augusto Sarti, Gregory Walsh and Shankar Sastry

Electronic Research Laboratory
Department of Electrical Engineering and Computer Sciences
University of California
Berkeley, CA 94720

Abstract

In this paper we generalize a technique for eliminating the drift from the description of a control system on a matrix Lie group with left-invariant vector fields. A diffeomorphism of the state space together with an affine input transformation are used in order to put the system into an equivalent left-invariant drift-free form. Techniques developed for steering drift-free control systems may then be applied. We apply this method to the Lie group of the rigid rotations $SO(3)$ as in the authors' previous work [6], and to a new example, the rigid motions $SE(3)$.

1 Introduction

Matrix Lie groups [1] often play important roles as configuration spaces in the study of mechanics [2, 3]. For example, the orientation of a rigid body can be represented by an element of the Lie group $SO(3)$, the space of the orthogonal matrices of determinant 1. The Lie algebra [4, 5] of $SO(3)$, denoted by so(3), is defined as the the set of all 3×3 skew–symmetric matrices, with the matrix commutator as the Lie bracket.

One other Lie group which arises often in the study of robotic systems is the group of the rigid motions $SE(3)$, defined as the subset:

$$SE(3) = \left\{ g \in I\!R^{4\times4} : g = \begin{bmatrix} R & r \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3), \quad r \in I\!R^3 \right\}$$

of the space of all 4×4 matrices, thus a 6-dimensional manifold. Its Lie algebra $se(3)$, can be identified with the space of all 4×4 matrices of the form

$$se(3) = \left\{ X \in I\!R^{4\times4} : X = \begin{bmatrix} V & v \\ 0 & 0 \end{bmatrix}, \quad V \in so(3), \quad v \in I\!R^3 \right\}$$

again with the matrix commutator as Lie bracket.

Both the examples above are Lie subgroups of $GL(n)$, the set of invertible $n \times n$ matrices [1]. The Lie algebra $gl(n)$ associated to the *general linear* group $GL(n)$, being the tangent space of $GL(n)$ at the identity, can be identified with the space of all $n \times n$ matrices with, once again, the matrix commutator as Lie bracket. We consider, in general, kinematic systems on matrix Lie groups which have left-invarient vector fields. We write their general form as follows:

$$\dot{g} = gX_0 + g\sum_{i=1}^{m} X_i u_i \quad , \tag{1}$$

where $g \in G$, G being the p-dimensional Lie subgroup of $GL(n)$ generated by the Lie algebra $\mathcal{G} \stackrel{\text{def}}{=} \overline{\text{span}}\{X_1, \ldots, X_m\}$, i.e. the involutive closure of the vector space spanned by the input generators.

There is a great deal of literature concerning the path planning problem, that is steering a drift-free control system from a given initial state to a given final state in an assigned time. In this paper we present a technique for eliminating the drift from the description of the system so that we may apply this existing literature. We will show that, by using a diffeomorphism of the state space together with an affine input transformation it is possible, under relatively weak conditions, to put the system into an equivalent left-invariant drift-free form. The generation of open-loop strategies that solve the steering problem for drift-free systems can then proceed.

We consider first the particular case of $SO(3)$ and derive the structure of both the state and the input transformations that put the system in drift-free form by using some specific and well-known properties of the considered group, as in the previous work of the authors [6]. After this introductory Section, we generalize this technique to the case of $GL(n)$ by means of the adjoint representation of the Lie group [4, 5]. The resulting method is shown to lead to the same results already found in the case of $SO(3)$. Furthermore, an example of application of the drift elimination method is given for the Lie group $SE(3)$.

2 Drift elimination from left-invariant control systems on $SO(3)$

Consider the system on $SO(3)$:

$$\dot{g} = gX_0 + gX_1u_1 + gX_2u_2\,, \quad X_1, X_2 \in so(3) \tag{2}$$

where X_1 and X_2 are linearly independent. It may be verified that $\mathcal{G} = \overline{\text{span}}\{X_1, X_2\} = so(3)$, therefore the drift-free system (2) is controllable by Chow's theorem [8]. The skew-symmetric matrices[1] X_0, X_1 and X_2 correspond to the vectors $b_0, b_1, b_2 \in I\!R^3$ respectively, that is, $X_i = (b_i\times)$, $i = 0, 1, 2$.

Proposition 1 *Given a system of the form (2), it is always possible to find an input transformation of the form*

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = B(t) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

such that

$$\dot{g}_r = g_rX_1v_1 + g_rX_2v_2\,, \tag{4}$$

where $g_r = g\exp(-X_0t)$.

[1] the skew symmetric version $(b\times)$ of a vector b, with components b^1, b^2 and b^3, is $(b\times) = \begin{bmatrix} 0 & -b^3 & b^2 \\ b^3 & 0 & -b^1 \\ -b^2 & b^1 & 0 \end{bmatrix}$

*Research supported in part by NSF under grant IRI90-14490

0191-2216/93/$3.00 © 1993 IEEE

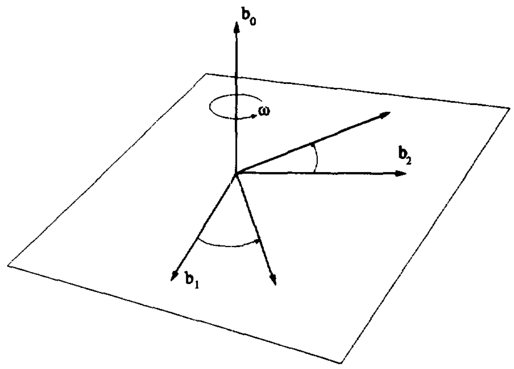

Figure 1: Note the plane Ω which the vectors b_1 and b_2 span remains constant under the action of the drift b_0 even though b_1 and b_2 do not.

In other words, the state diffeomorphism $g_r = g\exp(-X_0t)$ exactly compensates for the drift by making the moving frame rotate around b_0. The time-varying input transformation (3) takes into account the fact that the inputs are attached to the body while the moving frame is not, as shown in Figure 1.

Notice that the state space diffeomorphism $g_r = g\exp(-X_0t)$ is exactly the same as that used by Brockett to prove a controllability theorem [7] which is valid for systems defined on Lie groups that are generated by some commutative Lie subalgebras of $gl(n)$.

Proposition (1) may be proven in a constructive way.

Proof:

Without loss of generality, we assume b_0 to be orthogonal to both b_1 and b_2. In fact, if b_0 had a non zero component on the space spanned by b_1 and b_2, we could eliminate it by means of a linear combination of constant inputs. It is not restrictive, in addition, to assume that b_1 and b_2 are orthogonal and have unit norm, as this can be achieved by means of a linear input transformation of the inputs u_1 and u_2. Finally, denoting $||b_0||$ by ω, we assume that the orientation of the vectors b_0, b_1 and b_2 is such that $b_0 \times b_1 = \omega b_2$, $\omega b_1 \times b_2 = b_0$ and $b_2 \times b_0 = \omega b_1$. This may require that we relabel the inputs.

From $g_r = g\exp(-X_0t)$ we can write $g = g_r\exp(X_0t)$, whose derivative is

$$\dot{g} = \dot{g}_r e^{X_0t} + gX_0 \tag{5}$$

We want to find a new pair of inputs (v_1, v_2) such that

$$\dot{g}_r = g_r X_1 v_1 + g_r X_2 v_2 \tag{6}$$

The two inputs v_1 and v_2 that solve the problem can be determined by using equation (6) in equation (5) to obtain:

$$\dot{g} = ge^{-X_0t}X_1e^{X_0t}v_1 + ge^{-X_0t}X_2e^{X_0t}v_2 + gX_0 \tag{7}$$

We recall that, given any rotation matrix $g \in SO(3)$ and any skew symmetric matrix $(b\times) \in so(3)$, we have $g(b\times)g^{-1} = (gb\times)$, therefore equation (7) becomes

$$\dot{g} = gX_0 + g\,(c_1\times)\,v_1 + g\,(c_2\times)\,v_2 \tag{8}$$

where $c_1 = \exp(-X_0t)b_1$ and $c_2 = \exp(-X_0t)b_2$. These two terms can be computed by means of Rodrigues' formula

$$\begin{aligned} c_1 &= \left(I - X_0\frac{1}{\omega}\sin\omega t + X_0^2\frac{1}{\omega^2}(1-\cos\omega t)\right)b_1 \\ &= b_1 - b_0\times b_1\frac{1}{\omega}\sin\omega t + b_0\times(b_0\times b_1)\frac{1}{\omega^2}(1-\cos\omega t) \\ &= b_1\cos\omega t - b_2\sin\omega t\,, \end{aligned}$$

where I is the identity matrix. Similarly we have

$$c_2 = b_1\sin\omega t + b_2\cos\omega t\,.$$

Equation (8) thus becomes

$$\begin{aligned} \dot{g} &= g\,(X_1\cos\omega t - X_2\sin\omega t)\,v_1 + \\ &\quad g\,(X_1\sin\omega t + X_2\cos\omega t)\,v_2 + gX_0 \end{aligned} \tag{9}$$

It is now clear that, by setting

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = B(t)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{10}$$

we obtain system (2).

□

3 Drift elimination from left-invariant control systems on $GL(n)$

The above procedure for eliminating the drift from the description of a left-invariant control system defined on $SO(3)$ can be generalized to the case of the Lie group $GL(n)$ even though the tools involved in its derivation are valid only on $SO(3)$. One important technique in Section 2 is that, given any $(b\times) \in so(3)$ and any $g \in SO(3)$, we have $g(b\times)g^{-1} = (gb\times)$. This fact can be generalized to $GL(n)$ by using the adjoint representation of Lie groups.

3.1 Drift elimination using a time-varying input transformation

Theorem 1 *Let us consider the left-invariant system*

$$\dot{g} = gX_0 + g\sum_{i=1}^{m} X_i u_i\,, \tag{11}$$

where g belongs to the p-dimensional Lie subgroup G of $GL(n)$, generated by $\mathcal{G} \overset{\text{def}}{=} \overline{\text{span}}\{X_1,\ldots,X_m\}$.

Then, given a diffeomorphism of the state space of the form

$$g_r = ge^{-X_0t}\,, \tag{12}$$

there exists an input transformation of the form

$$u = B(t)v\,, \tag{13}$$

$B(t)$ being a smooth $m\times m$ matrix, $u = [u_1,\ldots,u_m]^T$ and $v = [v_1,\ldots,v_m]^T$, such that

$$\dot{g}_r = g_r\sum_{j=1}^{m} X_j v_j\,. \tag{14}$$

provided span$\{X_{m+1},\ldots,X_p\}$ *is closed under the action of the* adjoint operator[2] ad_{X_0}

[2]the adjoint operator is defined by $\text{ad}_{X_0}Y \overset{\text{def}}{=} [X_0, Y]$, $\forall Y \in \mathcal{G}$

The proof of Theorem 1 will be given in a constructive way. We will, in fact, show how to compute the time-varying input transformation matrix $B(t)$. Along the proof we will also show that the hypothesis on $\text{span}\{X_{m+1},\ldots,X_p\}$ can be replaced by a time-varying input constraint.

Proof:
As a first step we differentiate $g = g_r \exp(X_0 t)$, to obtain

$$\dot{g} = gX_0 + \dot{g}_r e^{X_0 t}\ , \qquad (15)$$

then we look for the conditions under which

$$\dot{g}_r = g_r \sum_{j=1}^{p} X_j v_j\ . \qquad (16)$$

Using equation (16) in (15) we obtain

$$\dot{g} = gX_0 + g\sum_{j=1}^{p} e^{-X_0 t} X_j e^{X_0 t} v_j \qquad (17)$$

We recall that the adjoint representation of a Lie group G [4, 5] is a map from $\mathcal{G}$ to $\mathcal{G}$ defined as

$$\text{Ad}g(X) \stackrel{\text{def}}{=} dK_g(X) \qquad X \in \mathcal{G},\quad g \in G,$$

$K_g : G \to G$ being the group conjugation map $K_g(h) = ghg^{-1}$. Since $G \subseteq GL(n)$, we have

$$\text{Ad}g(X) = gXg^{-1} \qquad g \in G,\quad X \in \mathcal{G}\ .$$

In general the adjoint representation can be reconstructed from its infinitesimal generators

$$\text{ad}_X Y \equiv \left.\frac{d}{dt}\right|_{t=0} \text{Ad}e^{Xt}(Y)\ , \qquad Y \in \mathcal{G}$$

through a Lie series of the form

$$\text{Ad}e^{X_0 t}(Y) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \text{ad}_{X_0}^k Y\ , \qquad (18)$$

where the infinitesimal adjoint action, for left-invariant vector fields, is found to agree at the identity with the Lie bracket[3] on $\mathcal{G}$

$$\text{ad}_X Y = [X, Y]$$

In fact we have

$$\begin{aligned} \left.\frac{d^k}{dt^k}\right|_{t=0} \text{Ad}e^{X_0 t}(X_j) &= \left.\frac{d^{k-1}}{dt^{k-1}}\right|_{t=0} \text{Ad}e^{X_0 t}(\text{ad}_{X_0} X_j) = \cdots \\ &= \text{Ad}e^{X_0 t}\left(\text{ad}_{X_0}^k X_j\right)\Big|_{t=0} \\ &= \text{ad}_{X_0}^k X_j\ . \end{aligned}$$

The Lie series (18) allows us to rewrite equation (17) as follows

$$\dot{g} = gX_0 + g\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{j=1}^{p} \text{ad}_{X_0}^k X_j v_j \qquad (19)$$

Let $\{X_1,\ldots,X_p\}$ be a basis for the Lie algebra $\mathcal{G}$. Then we can always write

$$\text{ad}_{X_0}^k X_j = \sum_{i=1}^{p} \alpha_{ij}^{(k)} X_i$$

[3] for the right-invariant vector fields the sign of the Lie bracket must be changed

therefore we have

$$\dot{g} = gX_0 + g\sum_{i=1}^{p} X_i \sum_{j=1}^{p} A_{ij}(t) v_j\ , \qquad (20)$$

where

$$A_{ij}(t) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \alpha_{ij}^{(k)}\ .$$

In order for system (20) to be equal to system (11) we need the new set of inputs $\overline{v} = [v_1,\ldots,v_p]^T$ to satisfy the following condition

$$\sum_{i=1}^{m} X_i u_i = \sum_{i=1}^{p} X_i \sum_{j=1}^{p} A_{ij}(t) v_j\ ,$$

which can be expressed in matrix form as follows

$$\overline{u} = A(t)\overline{v}\ , \qquad (21)$$

where $\overline{u} \stackrel{\text{def}}{=} [u_1,\ldots,u_m,0,\ldots,0]^T \in I\!R^p$ and $A(t)$ is a $p \times p$ matrix whose elements are given by $A_{ij}(t)$. Condition (21) can be split in two parts, the former being

$$u = A'(t)\overline{v} \qquad (22)$$

where $A'(t) \in I\!R^{m\times p}$ is the matrix given by the first m columns of $A(t)$, while the latter condition can be written as

$$\overline{v} \in \text{Ker}A''(t)\ . \qquad (23)$$

$A''(t) \in I\!R^{(p-m)\times p}$ being the matrix given by the last $p-m$ columns of $A(t)$. Equation (22) gives the relationship between the two sets of inputs u and $\overline{v}$ while equation (23) gives the constraint that $\overline{v}$ must satisfy.

The matching conditions (22) and (23) result as being greatly simplified when $\text{span}\{X_{m+1},\ldots,X_p\}$ is closed under the action of the adjoint operator ad_{X_0}, in which case $A(t)$ assumes the block-triangular form

$$A(t) = \begin{bmatrix} B(t) & C(t) \\ 0 & D(t) \end{bmatrix} \qquad (24)$$

where $B(t) \in I\!R^{m\times m}$. In this case, in fact, condition (23) becomes simply $v_i = 0$ for $i = m+1,\ldots,p$, therefore eq. (22) assumes the form (13).

□

Example 1: the case of $SO(3)$. We will now show how to obtain the results found in Section 2 by using Theorem 1.

Like in Section 2 we assume that the system has $m = 2$ inputs

$$\dot{g} = gX_0 + gX_1u_1 + gX_2u_2\ ,$$

where

$$[X_0, X_1] = \omega X_2\ , \quad \omega[X_1, X_2] = X_0\ , \quad [X_2, X_0] = \omega X_1\ .$$

Let $X_3 = [X_1, X_2]$ be the vector that we need to complete the basis for the Lie algebra $so(3)$. According to the above hypotheses, we have $X_0 = \omega X_3$.

The 3×3 matrices $\alpha^{(k)}$, whose entries are given by $\alpha_{ij}^{(k)}$, can be computed in a straightforward manner. The coefficients of the linear combination of the X_j's that gives $\text{ad}_{X_0}^k X_i$ are, in fact, the elements of the i^{th} row of the matrix $\alpha^{(k)}$. For example, considering that $\text{ad}_{X_0}^0 X_1 = X_1$, we have $\alpha_{11}^{(0)} = 1$, $\alpha_{12}^{(0)} = 0$ and $\alpha_{13}^{(0)} = 0$.

A direct calculation will give $\alpha^{(0)} = I$, I being the identity matrix, and

$$\alpha^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \alpha^{(2)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha^{(3)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \alpha^{(4)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \cdots$$

Notice that, since $[X_0, X_3] = 0$, span$\{X_3\}$ results as being closed under the action of adX_0, therefore we may expect $A(t)$ to assume the block-triangular form (24). In fact it results

$$A(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\alpha^{(k)} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

therefore condition (23) can be expressed as $v_3 = 0$ and the input transformation (13) assumes the form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

□

3.2 Drift elimination using an affine input transformation

The condition on span$\{X_{m+1}, \ldots, X_p\}$ in Theorem 1 can be made less restrictive by using an *affine input transformation* rather than a time-varying input transformation of the form (13).

Theorem 2 *Let us consider the left-invariant system*

$$\dot{g} = gX_0 + g\sum_{i=1}^{m} X_i u_i \tag{25}$$

where g belongs to the p-dimensional Lie subgroup G of $GL(n)$, generated by $\mathcal{G} \stackrel{\text{def}}{=} \overline{\text{span}}\{X_1, \ldots, X_m\}$.

Then, there exists an affine input transformation of the form

$$u = B(t)v + U\ , \tag{26}$$

where $u = [u_1, \ldots, u_m]^T$, $v = [v_1, \ldots, v_m]^T$, $U \in \mathbb{R}^m$ is a constant vector and $B(t)$ is a smooth $m \times m$ matrix, and a diffeomorphism of the state space of the form

$$g_r = ge^{-Y_0 t}\quad , \tag{27}$$

where

$$Y_0 \stackrel{\text{def}}{=} X_0 + \sum_{i=1}^{m} U_i X_i \tag{28}$$

such that

$$\dot{g}_r = g_r \sum_{j=1}^{m} X_j v_j \tag{29}$$

if $\text{ad}_{X_0}X_j \in \text{span}\{X_{m+1}, \ldots, X_p, \text{ad}_{X_1}X_j, \ldots, \text{ad}_{X_m}X_j\}$, *for* $j = m+1, \ldots, p$.

Proof:

Applying transformation (28) to system (25) is equivalent to applying the input transformation

$$w = B(t)v\ , \tag{30}$$

where $w \in \mathbb{R}^m$, to the system

$$\dot{g} = gY_0 + g\sum_{i=1}^{m} X_i w_i\ , \tag{31}$$

where the drift term Y_0 is now given by equation (28). But we already know from Theorem 1 that there exists an input transformation of the form (30) that puts system (31) in the drift-free form (29) through the diffeomorphism of the state space (27) provided that

$$\text{ad}_{Y_0}X_j \in \text{span}\{X_{m+1}, ..., X_n\}\ , \quad j = m+1, \ldots, p \tag{32}$$

By using eq. (28) we can also write

$$\text{ad}_{Y_0}X_j = \text{ad}_{X_0}X_j + \sum_{i=1}^{m} U_i \text{ad}_{X_i}X_j\ , \quad j = 1, \ldots, p\ ,$$

which allows us to write condition (32) as the condition on the input generators of Theorem 2.

□

By adding a constant term we substantially broaden the range of possible systems that can be made drift-free in their description. This will result as clear in the next examples.

Example 2: the case of $SO(3)$. In Example 1 the orthogonality between X_0 and the vector space spanned by X_1 and X_2 was necessary condition for the existence of an input transformation of the form (13) that makes the system drift-free. On the other hand, in Section 2 we pointed out that, if X_0 had a non zero component on the space spanned by X_1 and X_2, it would be easy to cancel for it by means of a linear combination of constant inputs. By allowing the input transformation to be in affine form we essentially generalize this orthogonalization procedure to the case of $GL(n)$.

Let

$$\dot{g} = gX_0 + gX_1 v_1 + gX_2 v_2$$

be a left-invariant system defined on $SO(3)$, where $X_0, X_1, X_2 \in so(3)$ are assumed to be independent. Let us assume X_1 and X_2 to be *orthogonal* and have unit norm, which is to say

$$< X_1, X_2 > \stackrel{\text{def}}{=} \frac{1}{2}\text{Tr}\{X_1^T X_2\} = 0$$

and

$$\|X_i\| = \left\{\frac{1}{2}\text{Tr}\{X_i^T X_i\}\right\}^{\frac{1}{2}} = 1\ , \quad i = 1, 2\ .$$

Such conditions are not restrictive as they can be easily achieved through linear combination of the inputs.

If we use an affine input transformation of the form (26), where the constants are chosen as $U_1 = - < X_0, X_1 >$ and $U_2 = - < X_0, X_2 >$, we essentially *orthogonalize* the action of the drift with respect of the input generators. In fact we obtain a system of the form (31), which satisfies the following conditions

$$[Y_0, X_1] = \omega X_2\ , \quad \omega[X_1, X_2] = X_0\ , \quad [X_2, Y_0] = \omega X_1\ ,$$

we thus are again in the situation of Example 1.

This orthogonalization procedure is always possible as the condition on the input generators of Theorem 2

$$\text{ad}_{X_0}X_3 \in \text{span}\{X_3, [X_1, X_3], [X_2, X_3]\} \equiv \mathcal{G}$$

is always satisfied.

□

Example 3: the case of $SE(3)$ Let us consider a system of the form

$$\dot{g} = gX_0 + gX_1u_1 + gX_2u_2 + gX_3u_3 \ ,$$

where

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The involutive closure of the vector space spanned by the input generators is $\mathcal{G} = \text{span}\{X_1, \ldots, X_6\} = se(3)$, where $X_4 \stackrel{\text{def}}{=} [X_1, X_2]$, $X_5 \stackrel{\text{def}}{=} [X_3, X_1]$ and $X_6 \stackrel{\text{def}}{=} [X_2, X_3]$ are given by

$$X_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case the condition on the input generators of Theorem 2 is given by

$$[X_0, X_4] \in \text{span}\{X_1, X_2, X_4, X_5, X_6\} \quad (33)$$
$$[X_0, X_5] \in \text{span}\{X_3, X_4, X_5, X_6\} \quad (34)$$
$$[X_0, X_6] \in \text{span}\{X_4, X_5, X_6\} \ . \quad (35)$$

For an arbitrary drift vector X_0 of the form

$$X_0 = \begin{bmatrix} 0 & -V_3 & V_2 & v_1 \\ V_3 & 0 & -V_1 & v_2 \\ -V_2 & V_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it is easy to check that conditions (33,34) are always satisfied, while condition (35) is satisfied when $V_2 = 0$. In other words the system can be put in an equivalent drift-free form if X_0 is of the form

$$X_0 = \begin{bmatrix} 0 & -V_3 & 0 & v_1 \\ V_3 & 0 & -V_1 & v_2 \\ 0 & V_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If, for example, $X_0 = X_3$, we obtain $g_r = g(I - X_0t)$ and $B(t) = I$. In this case, in fact, X_0 can be easily canceled by adding appropriate constants to the inputs. On the other hand, if $X_0 = X_6$, then we have again $g_r = g(I - X_0t)$ but

$$B(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \ .$$

□

4 Conclusion

In this paper we proposed a technique for eliminating the drift term from the description of a left-invariant control system on the general matrix Lie group $GL(n)$. We have shown that, by using a diffeomorphism of the state space together with an affine input transformation, it is possible to put the system in an equivalent left-invariant drift-free form. We have determined the conditions under which this method is applicable and expressed in a closed form both the state and the input transformation that lead to the desired result. Examples of application of the above method have been developed for the Lie groups of the rigid rotations $SO(3)$ and of the rigid motions $SE(3)$.

References

[1] M. L. Curtis. *Matrix Groups.* Second edition. Springer–Verlag, Berlin, Heidelberg 1984.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York, New York, second edition, 1989.

[3] J. Marsden. *London Mathematical Society Lecture Note Series 174: Lectures on Mechanics.* Cambridge University Press, Cambridge, England, 1992.

[4] P. J. Olver. *Applications of Lie Groups to Differential Equations.* Graduate texts in mathematics No. 107. Springer-Verlag, Berlin, Heidelberg 1986.

[5] V. S. Varadarajan. *Lie Groups, Lie Algebras and their Representations.* Graduate texts in mathematics No. 102. Springer-Verlag, Berlin, Heidelberg 1984.

[6] G. Walsh, A. Sarti and S. Sastry. Algorithms for steering on the group of rotations. In *1993 American Control Conference*, San Francisco, California, June 2-4, 1993.

[7] R. W. Brockett. System Theory on Group Manifolds and Coset Spaces. *SIAM J. Control*, Vol. 10, No. 2, May 1972, pages 265-284.

[8] W.L. Chow. Über systeme von linearen partiallen differentialgleichungen erster ordnung. *Math. Annalen*, 117:98-105, 1940-41.

[9] H. Nijmeijer, A. Van Der Schaaft. *Nonlinear Dynamical Control Systems.* Springer-Verlag, Berlin, Heidelberg 1990.