Communication Theory

Recursive Techniques for the Synthesis of a p^{th}-Order Inverse of a Volterra System

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Abstract. A new recursive technique for the synthesis of a p^{th}-order inverse of a Volterra system is presented. In such a method, the condition that the p^{th}-order inverse be exactly of order p, as proposed by Schetzen, is relaxed. The choice of the residuals, that is the operators of the inverse whose order is higher than p, is done with the purpose of reducing the complexity of the synthesis scheme and of deriving a recursive procedure to build-up such an inverse.

A comparison between the complexity of Schetzen's synthesis schemes and those obtained with our recursive procedure shows that the structures that we obtain are much less complex and easier to derive, and the increment of complexity related to the increase in order is much slower than in Schetzen's case.

As an example of application, we use the proposed method in order to linearize a digital radio link in which the high-power amplifier is operated near saturation.

1. INTRODUCTION

Functional expansions are widely used in system theory to obtain mathematical representations of nonlinear systems. Although the functional series were presented for the first time in 1887 by the mathematician V. Volterra [1, 2], the researcher who first introduced this representation into nonlinear circuit analysis was N. Wiener [3], whose merit was primarily that of using the Volterra series in order to characterize the response of a nonlinear device. After his early work, Wiener devoted his efforts to many problems of analysis and synthesis of nonlinear systems [4]. Wiener's work was followed by many formal studies of the application of the Volterra series to nonlinear systems, among which we recall those of Barrett [5], Brilliant [6] and George [7]. Other important studies concerning the operational representation were due to Zames [8] and, more recently, to Sandberg [9, 10]. The extension to the discrete systems of a Volterra series was first proposed and studied by Alper [11] in 1964.

An important problem which has involved many researches for quite some time is that of the inversion of a Volterra system. There are several possible applications of such an aspect of the system theory. Most of them are related with the problem of the linearization of systems, but there are also many other cases of interest among which we recall the solution of nonlinear differential equations and the operatorial representation of nonlinear feedback systems.

A crucial aspect of the inversion of a Volterra system is the determination of the conditions under which the system admits inverse. It is well known that not all the nonlinear systems possess an inverse and that many systems possess an inverse only for a restricted range of the input amplitude. On the other hand there is a class of inverses, called p^{th}-order inverses, in which the input amplitude range is not restricted. The theory of the p^{th}-order inverses was developed by Schetzen [12, 13] in 1976. He obtained the necessary and sufficient conditions for the existence of the p^{th}-order inverse and proposed a method for its synthesis. The resulting structure of such a method is a p^{th}-order Volterra system, whose structural complexity increases very rapidly with the order.

In this paper we propose a new approach to the synthesis, relaxing the condition that the p^{th}-order inverse be of order p, i.e. allowing it to have kernels of higher order as well. In this way we can simplify the synthesis procedure and reduce the resulting structural complexity. We illustrate some properties of the elementary interconnections of Volterra systems, that is the *parallel* and the *cascade,* and introduce some simple rules that allow us to better understand the behavior of such interconnections. With these tools we derive a recursive technique for the synthesis of p^{th}-order preinverses and postinverses. Finally we prove that, even if it has kernels whose order is higher than p, a p^{th}-order preinverse is also a p^{th}-order postinverse.

The material presented in this paper is organized in such a way as to provide the reader with all the tools necessary to understand it. In particular, Section 2 is devoted to a brief introduction to the Volterra systems and their operatorial representation. In Section 3 we discuss the properties of the interconnection which are necessary for the inversion problem. The recursive techniques for the synthesis of preinverses and postinverses are developed and compared with the Schetzen's method in Section 4. Finally, in Section 5 an example of application to the digital radio transmission systems is considered and some simulation results are reported and discussed.

2. THE VOLTERRA OPERATORIAL REPRESENTATION

A wide class of nonlinear systems with memory can be modeled by means of an operatorial series

$$y(t) = \boldsymbol{H}[x(t)] = \sum_{n=1}^{\infty} \boldsymbol{H}_n[x(t)] \tag{1}$$

where $\boldsymbol{H}$ represents the system operator and $\boldsymbol{H}_n$ is the n^{th}-order operator, defined as one for which the response to a linear combination of signals is an n-linear operation on the individual signals

$$\boldsymbol{H}_n\left[\sum_{k=1}^{N} c_k x_k(t)\right] = \sum_{k_1=1}^{N} \dots \sum_{k_n=1}^{N} c_{k_1} \dots$$
$$\dots c_{k_n} \boldsymbol{H}_n\{x_{k_1}(t), \dots, x_{k_n}(t)\} \tag{2}$$

where c_k, $k = 1, \dots, n$, are arbitrary constants and $\boldsymbol{H}_n\{x_1(t), \dots, x_n(t)\}$ is an n-linear operator (linear in each argument when the others are held fixed).

The functional characterization of the system $\boldsymbol{H}$ is the Volterra series, in which the n^{th}-order Volterra operator is described by the I/O relationship

$$\boldsymbol{H}_n[x(t)] = \int_{\mathcal{G}^n} h_n(\tau_1, \dots, \tau_n) \prod_{j=1}^{n} x(t-\tau_j)\, d\tau_j \tag{3}$$

where $h_n(\tau_1, \dots, \tau_n)$ is the n^{th}-order Volterra kernel, and the n-linear operator of $\boldsymbol{H}$ is

$$\boldsymbol{H}_n\{x_1(t), \dots, x_n(t)\} =$$
$$= \int_{\mathcal{G}^n} h_n(\tau_1, \dots, \tau_n) \prod_{i=1}^{n} x_i(t-\tau_i)\, d\tau_i \tag{4}$$

Eq. (3) represents, for the continuous domain $\mathcal{G} = \mathcal{R}$, a multiple convolution integral

$$\boldsymbol{H}_n[x(t)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n) x(t-\tau_1) \dots$$
$$\dots x(t-\tau_n)\, d\tau_1 \dots d\tau_n \tag{5}$$

and, when $\mathcal{G}$ is the discrete domain $\mathcal{Z}(T)$ (a domain whose samples are equally spaced by the quantum T), it represents a multiple sum

$$\boldsymbol{H}_n[x(nT)] =$$
$$= \sum_{k_1, \dots, k_n} h_n(k_1 T, \dots, k_n T) \prod_{j=1}^{n} Tx(nT - k_j T) \tag{6}$$

obtained by eq. (3) by means of the substitutions $t = nT$, $\tau_j = k_j T$ and $d\tau_j = T$.

In this paper we consider the class of bounded input, bounded output (BIBO) stable nonlinear systems whose system operator can be expressed in the power series with memory (1).

In general a functional series, seen as an I/O relationship, is rather difficult to handle, expecially when we are going to analyze structures which are more complex than a simple cascade or a simple feedback. In particular the problem of the inversion of a Volterra system could be very difficult or even impractical to afford, as such, an operator-based approach is much more convenient. This fact will be shown in next Section.

3. INTERCONNECTION OF VOLTERRA SYSTEMS

Before presenting the theory of the recursive synthesis of a p^{th}-order inverse we briefly discuss some properties of the interconnections between operators (system operators and Volterra operators). In particular we consider a pair of elementary interconnections which we refer to as *parallel* and *cascade*.

3.A. *Elementary interconnections*

The *parallel* of two systems $\boldsymbol{H}$ and $\boldsymbol{K}$, referred to as $\boldsymbol{H} \hat{+} \boldsymbol{K}$, is a Volterra system such that

$$(\boldsymbol{H} \hat{+} \boldsymbol{K})[x] = \sum_{n=1}^{\infty} (\boldsymbol{H}_n[x] + \boldsymbol{K}_n[x]) = \boldsymbol{H}[x] + \boldsymbol{K}[x] \tag{7}$$

because the n^{th}-order kernel of the parallel is the sum of the n^{th}-order kernels of the systems $\boldsymbol{H}$ and $\boldsymbol{K}$.

The characterization of the *cascade* of two systems $\boldsymbol{H}$ and $\boldsymbol{K}$, denoted by $\boldsymbol{H} \circ \boldsymbol{K}$, is a system such that:

$$\boldsymbol{H} \circ \boldsymbol{K}[x] = \boldsymbol{K}[\boldsymbol{H}[x]] = \sum_{n=1}^{\infty} \boldsymbol{K}_n\left[\sum_{m=1}^{\infty} \boldsymbol{H}_m[x]\right]$$
$$= \sum_{n=1}^{\infty} \sum_{m_1=1}^{\infty} \dots \sum_{m_n=1}^{\infty} \boldsymbol{K}_n\{\boldsymbol{H}_{m_1}[x], \dots, \boldsymbol{H}_{m_n}[x]\} \tag{8}$$

$\boldsymbol{K}_n\{x_1, \dots, x_n\}$ being the n-linear operator of $\boldsymbol{K}$.

In eq. (8), the single Volterra operators of the cascade are not distinguishable from each other, therefore such an expression is not written as a Volterra series. Noticing that $\boldsymbol{K}_n\{\boldsymbol{H}_{m_1}[x], \dots, \boldsymbol{H}_{m_n}[x]\}$ can be seen as the response of a Volterra operator of order $m_1 + \dots + m_n$ driven by the excitation x, we can write the k^{th}-order term of the eq. (8) as

$$\boldsymbol{Q}_k[x] = \sum_{n \ge 1} \underbrace{\sum_{m_1 \ge 1} \dots \sum_{m_n \ge 1}}_{m_1 + \dots + m_n = k} \boldsymbol{K}_n\{\boldsymbol{H}_{m_1}[x], \dots, \boldsymbol{H}_{m_n}[x]\} \tag{9}$$

where the range of the summation index n can be limited to $1 \le n \le k$ and the constraints on $m_1, \dots, m_n$, ensure that $1 \le m_1, \dots, m_n \le k - n + 1$.

Eq. (9) gives the k^{th}-order Volterra operator of $\boldsymbol{Q}$ in terms of the Volterra operators of $\boldsymbol{H}$ and the multilinear operators of $\boldsymbol{K}$, therefore it has not any structural meaning. In other words, it does not represent an interconnection of Volterra operators of $\boldsymbol{H}$ and $\boldsymbol{K}$.

A method for rewriting eq. (9) as an interconnection of Volterra operators is described in Schetzen's work [12, 13] and represents the starting point of his technique for the synthesis of the p^{th}-order inverse. Our procedure for the synthesis is based on a different approach which, in order to be explained, needs to be preceded by a discussion on some simple properties of the interconnections of operators.

3.B. *Some properties of the interconnections*

In the following considerations we will refer to the p^{th}-order truncation of a system operator $\boldsymbol{H}$ as

$$\boldsymbol{H}_{(p)} = \left[\sum_{n=1}^{\infty} \boldsymbol{H}_n\right]_{(p)} = \sum_{n=1}^{\infty} \boldsymbol{H}_n \tag{10}$$

where the argument of the operators has been omitted and the summation of operators is intended as a multiple parallel interconnection. Moreover we will refer to the p^{th}-order Volterra operator of a system as

$$\left[\sum_{n=1}^{\infty} \boldsymbol{H}_n\right]_p = \boldsymbol{H}_p \tag{11}$$

With the above notation it is easy to introduce three simple properties, to be used in the next Section, which can be easily verified by means of eqs. (9), (10) and (11).

Property 1. *Given a pair of Volterra operators $\boldsymbol{H}_m$ and $\boldsymbol{K}_n$ of order m and n, respectively, we have*

$$[\boldsymbol{H}_m \circ \mathrm{K}_n]_k = \begin{cases} \boldsymbol{H}_m \circ \boldsymbol{K}_n & \text{if } mn = k \\ \boldsymbol{0} & \text{otherwise,} \end{cases} \tag{12}$$

where $\boldsymbol{0}$ is the *zero* system operator, which maps every input function in the identically zero function. As a consequence of Property 1 we have

Property 2. *Let $\boldsymbol{H}_m$ be a Volterra operator and $\boldsymbol{K}$ a system operator. The k^{th}-order truncation of their cascade, with $k \geq m$, is*

$$[\boldsymbol{H}_m \circ \boldsymbol{K}]_{(k)} = \boldsymbol{H}_m \circ \boldsymbol{K}_{(n)} \tag{13}$$

where $n = \lfloor k/m \rfloor$ is the greatest integer not greater than k/m.

When the Volterra operator follows the system operator we have

Property 3. *Let $\boldsymbol{H}$ be a system operator and $\boldsymbol{K}_n$ be an n^{th}-order Volterra operator. The k^{th}-order Volterra operator of their cascade, with $k \geq m$, is*

$$[\boldsymbol{H} \circ \boldsymbol{K}_n]_k = [\boldsymbol{H}_{(k-n+1)} \circ \boldsymbol{K}_n]_{\mathrm{k}} \tag{14}$$

Property 3 states that the operators $\boldsymbol{H}_i$, $i > k-n+1$, give no contribution to the k^{th}-order kernel of $\boldsymbol{H} \circ \boldsymbol{K}_n$.

4. p^{th}-ORDER INVERSION

The p^{th}-order inverse of a nonlinear system $\boldsymbol{Q}$ is commonly referred to as a p^{th}-order system $\boldsymbol{Q}_{(p)}^{-1}$ such that, cascaded with $\boldsymbol{Q}$, results in a system in which the first-order operator is $\boldsymbol{I}$ ($\boldsymbol{I}$ being the *identity* operator, which maps every input function in itself), and the second through the p^{th}-order Volterra kernels are zero.

Hereafter we use a slightly different definition of such an inverse, which gives us a certain degree of freedom and allows us to follow a new approach for its synthesis.

Definition 1. *the system $\boldsymbol{P}^{(p)}$ is a p^{th}-order preinverse of $\boldsymbol{Q}$ if*

$$[\boldsymbol{P}^{(p)} \circ \boldsymbol{Q}]_{(p)} = \boldsymbol{I} \tag{15}$$

whatever the order of the system $\boldsymbol{P}^{(p)}$ may be.

The postinverse is defined as in Definition 1, apart from a change in the order of connection of the operators.

Definition 1 emphasizes that the p^{th}-order inverse is a system whose order is allowed to be greater than or equal to p. This means that its general expression is

$$\boldsymbol{P}^{(p)} = \sum_{k=1}^{p} \boldsymbol{P}_k \hat{+} \sum_{k=p+1}^{\infty} \boldsymbol{R}_k^{(p)} \tag{16}$$

where the operators $\boldsymbol{P}_k$, $1 \leq k \leq p$, are determined by the conditions to be imposed on the first p Volterra operators of the cascade of $\boldsymbol{P}^{(p)}$ and $\boldsymbol{Q}$, while $\boldsymbol{R}_k^{(p)}$, $k > p$, are unconstrained residual operators whose choice represents the degree of freedom that we have in the design of the inverse. The residuals $\boldsymbol{R}_k^{(p)}$ could be chosen in order to minimize the complexity of the resulting structure, or some other parameters of interest, and to preserve the conditions of BIBO stability for $\boldsymbol{P}^{(p)}$. In the following paragraphs we will show two syntheses of $\boldsymbol{P}^{(p)}$ that guarantee a BIBO stability.

4.A. *p^{th}-Order preinversion*

Let us consider the cascade of two systems $\boldsymbol{P}$ and $\boldsymbol{Q}$. In order to be a p^{th}-order preinverse, $\boldsymbol{P}$ must satisfy two conditions which represent a different way of writing eq. (15)

$$[\boldsymbol{P} \circ \boldsymbol{Q}]_1 = \boldsymbol{I} \tag{17}$$

$$[\boldsymbol{P} \circ \boldsymbol{Q}]_k = \boldsymbol{0}, \quad 2 \leq k \leq p \tag{18}$$

Properties 2 and 3 allow us to write the k^{th}-order Volterra operator of $\boldsymbol{P} \circ \boldsymbol{Q}$ as

$$\begin{aligned} [\boldsymbol{P} \circ \boldsymbol{Q}]_k &= \left[\left(\sum_{i=1}^{\infty} \boldsymbol{P}_i\right) \circ \sum_{j=1}^{\infty} \boldsymbol{Q}_j\right]_k \\ &= \sum_{j=1}^{\infty} \left[\left(\sum_{i=1}^{\infty} \boldsymbol{P}_i\right) \circ \boldsymbol{Q}_j\right]_k \\ &= \sum_{j=1}^{k} \left[\left(\sum_{i=1}^{k-j+1} \boldsymbol{P}_i\right) \circ \boldsymbol{Q}_j\right]_k \end{aligned} \tag{19}$$

therefore, for $k = 1$, we get

$$[\boldsymbol{P} \circ \boldsymbol{Q}]_1 = \boldsymbol{P}_1 \circ \boldsymbol{Q}_1 \tag{20}$$

which means that the first-order Volterra operator of the cascade is the cascade of the first-order Volterra operators of the involved systems, therefore condition (17) becomes

$$\boldsymbol{P}_1 \circ \boldsymbol{Q}_1 = \boldsymbol{I} \tag{21}$$

which requires $\boldsymbol{P}_1$ to be the inverse operator of $\boldsymbol{Q}_1$. Notice that it is not always possible to find a BIBO stable and causal linear operator $\boldsymbol{P}_1$ which satisfies the condition (21), therefore its existence is a necessary condition for the solution of the synthesis problem. Under this hypothesis, condition (21) becomes

$$P_1 = Q_1^{-1} \tag{22}$$

Eq. (19) allows us to rewrite the condition (18) as follows

$$P_k \circ Q_1 \hat{+} \sum_{j=2}^{k} \left[\left(\sum_{i=1}^{k-j+1} P_i \right) \circ Q_j \right]_k = \mathbf{0}, \quad 2 \le k \le p \tag{23}$$

With some simple manipulations on both sides of the operatorial equation (23), we get the following expression of the first p Volterra operators of the system P

$$P_k = - \left\{ \sum_{j=2}^{k} \left[\left(\sum_{i=1}^{k-j+1} P_i \right) \circ Q_j \right]_k \right\} \circ Q_1^{-1}$$
$$= \sum_{j=2}^{k} \left[\left(\sum_{i=1}^{k-j+1} P_i \right) \circ Q_j \right]_k \circ (-P_1), \quad 2 \le k \le p \tag{24}$$

where inserting a *minus* sign before an operator means changing the sign of its response or changing the sign of the response of the linear operatorP_1 that follows.

It is now possible to write the general expression of the p^{th}-order preinverse $P^{(p)}$ substituting eqs. (22) and (24) into eq. (16)

$$P^{(p)} = P_1 \hat{+} \sum_{k=2}^{p} \sum_{j=2}^{k} \left[\left(\sum_{i=1}^{k-j+1} P_i \right) \circ Q_j \right]_k \circ (-P_1) \hat{+} \sum_{k=p+1}^{\infty} R_k^{(p)}$$
$$= P_1 \hat{+} \sum_{k=2}^{p} \sum_{j=2}^{k} \left[\left(\sum_{i=1}^{k-j+1} P_i \hat{+} \sum_{i=k-j+2}^{p-1} P_i \hat{+} \right.\right.$$
$$\left.\left. \hat{+} \sum_{i=p}^{\infty} R_i^{(p-1)} \right) \circ Q_j \right]_k \circ (-P_1) \hat{+} \sum_{k=p+1}^{\infty} R_k^{(p)}$$
$$= P_1 \hat{+} \sum_{k=2}^{p} \sum_{j=2}^{k} [P^{(p-1)} \circ Q_j]_k \circ (-P_1) \hat{+} \sum_{k=p+1}^{\infty} R_k^{(p)}$$
$$= P_1 \hat{+} \sum_{k=2}^{p} [P^{(p-1)} \circ Q'_{(p)}]_k \circ (-P_1) \hat{+} \sum_{k=p+1}^{\infty} R_k^{(p)} \tag{25}$$

where $Q'_{(p)}$ is the p^{th}-order truncation of the system Q in which the linear operator Q_1 has been suppressed

$$Q'_{(p)} = \sum_{j=2}^{p} Q_j \tag{26}$$

and, according to Property 3, the second and the third summations of

$$P^{(p-1)} = \sum_{i=1}^{k-j+1} P_i \hat{+} \sum_{i=k-j+2}^{p-1} P_i \hat{+} \sum_{i=p}^{\infty} R_i^{(p-1)} \tag{27}$$

which have been inserted in eq. (25), give no contribution to $P^{(p)}$.

By making the following choice of the residuals:

$$R_k^{(p)} = [P^{(p-1)} \circ Q'_{(p)}]_k \circ (-P_1), \qquad k > p \tag{28}$$

eq. (25) becomes

$$P^{(p)} = P_1 \hat{+} \sum_{k=2}^{\infty} [P^{(p-1)} \circ Q'_{(p)}]_k \circ (-P_1)$$
$$= P_1 \hat{+} P^{(p-1)} \circ Q'_{(p)} \circ (-P_1) \tag{29}$$

Therefore $P^{(p)}$ can be synthesized by means of the following recursive operatorial equation

$$P^{(p)} = (P^{(p-1)} \circ Q'_{(p)} \hat{+} (-I)) \circ (-P_1) \tag{30}$$

whose structure, shown in Fig. 1, holds if and only if eq. (21) is satisfied by a BIBO stable and causal linear operator $P_1 = Q_1^{-1}$.

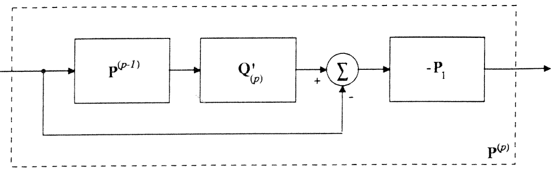

Fig. 1 - First recursive scheme of synthesis.

4.B. *A different scheme of synthesis*

According to Definition 1, the p^{th}-order preinverses of a system Q differ from each other in their residual kernels $R_k^{(p)}$, $k > p$. For this reason eq. (30) leads to only one of the possible classes of preinverses. Such a class is characterized by eq. (28). Other schemes of synthesis are possible with different choices of residuals. For example, applying Properties 2 and 3 to eq. (30) we get

$$[P^{(p)}]_{(p)} = [(P^{(p-1)} \circ Q'_{(p)} \hat{+} (-I))]_{(p)}$$
$$= \left[\left(\sum_{j=2}^{p} P^{(p-j+1)} \circ Q_j \hat{+} (-I) \right) \circ (-P_1) \right]_{(p)} \tag{31}$$

therefore it is sufficient to choose

$$R_k^{(p)} = \left[\left(\sum_{j=2}^{p} P^{(p-j+1)} \circ Q_j \hat{+} (-I) \right) \circ (-P_1) \right]_k, \quad k > p \tag{32}$$

to get a scheme of synthesis which is different from the previous one

$$P^{(p)} = \left(\sum_{j=2}^{p} P^{(p-j+1)} \circ Q_j \hat{+} (-I) \right) \circ (-P_1) \tag{33}$$

The structure corresponding to eq. (33), depicted in Fig. 2, seems to be more complex than that of Fig. 1, because it involves all the preinverses whose order is lower than p. On the other hand, such a complexity increment is only apparent because the degree of parallelization is much greater than in the other case.

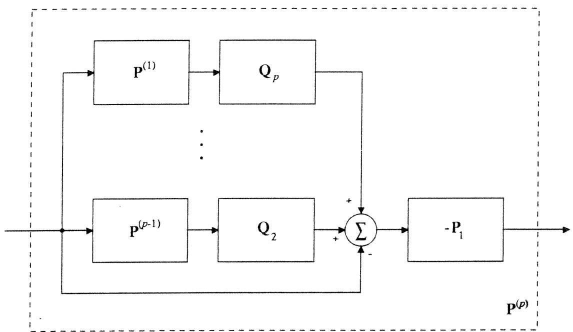

Fig. 2 - Second recursive scheme of synthesis.

4.C. p^{th}-Order postinversion

We prove that a p^{th}-order preinverse is also a p^{th}-order postinverse in the case in which residual Volterra operators are present. This result has been proven by Schetzen [12, 13] in the less general situation in which the inverse is exactly a p^{th}-order system.

Theorem 1. *Let Q be a system operator and $P^{(p)}$ be a system of any order such that*

$$[P^{(p)} \circ Q]_k = \begin{cases} I, & k = 1 \\ 0, & 2 \le k \le p \end{cases} \tag{34}$$

then we have

$$[Q \circ P^{(p)}]_k = \begin{cases} I, & k = 1 \\ 0, & 2 \le k \le p \end{cases} \tag{35}$$

Proof: it is made by induction.

Using eq. (34) and Property 3 we get

$$[P^{(p)} \circ Q \circ P^{(p)}]_k = \left[\left(I \hat{+} \sum_{i=p+1}^{\infty} S_i^{(p)}\right) \circ P^{(p)}\right]_k$$

$$= \left[\left[I \hat{+} \sum_{i=p+1}^{\infty} S_i^{(p)}\right]_k \circ P^{(p)}\right]_k = P_k, \; k \le p \tag{36}$$

which, for $k = 1$, gives

$$P_1 \circ Q_1 \circ P_1 = P_1 \tag{37}$$

therefore we have

$$Q_1 \circ P_1 = I \tag{38}$$

Letting $H = Q \circ P^{(p)}$ we have $H_2 = 0$. In fact

$$[P^{(p)} \circ Q \circ P^{(p)}]_2 = [P^{(p)} \circ (I \hat{+} H_2)]_2 = P_2 \hat{+} P_1 \circ H_2 \tag{39}$$

Recalling eq. (36) it results $P_1 \circ H_2 = 0$, therefore we have $H_2 = 0$ because, in general, P_1 is not a *zero* operator.

Finally, supposing that $H_2 = H_3 = \ldots = H_{k-1} = 0$, we get $H_k = 0$ because

$$[P^{(p)} \circ Q \circ P^{(p)}]_k = \left[P^{(p)} \circ \sum_{i=1}^{k} H_i\right]_k$$

$$= [P^{(p)} \circ (I \hat{+} H_k)]_k = P_k \hat{+} P_1 \circ H_k \tag{40}$$

As done for H_2, a simple comparison between eqs. (36) and (40) gives $P_1 \circ H_k = 0$, therefore it results $H_k = [Q \circ P^{(p)}]_k = 0$, $2 \le k \le p$, as stated by the Theorem.

4.D. Comparison between synthesis strategies

A comparison between inverses of the same order resulting from the application of our recursive technique and Schetzen's method [12, 13] can be easily made. In the following, Schetzen's p^{th}-order inverses will be called *minimal* inverses, since their order is the minimum allowed p.

Eqs. (30) and (33), as shown in Fig. 3, give rise to the same second-order inverse

$$P^{(2)} = P_1 \circ (Q_2 \circ (-P_1) \hat{+} I) \tag{41}$$

which, incidentally, results as minimal (without residual operators). In fact Schetzen's method yields the same result.

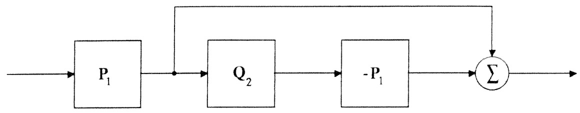

Fig. 3 - Second-order inverse.

Third-order inverses resulting from the application of eqs. (30) and (33) have different structures. The former results

$$P^{(3)} = P_1 \circ \{(Q_2 \circ (-P_1) \hat{+} I) \circ (Q_2 \hat{+} Q_3) \circ (-P_1) \hat{+} I\} \tag{42}$$

and the latter is given by

$$P^{(3)} = P_1 \circ \{((Q_2 \circ (-P_1) \hat{+} I) \circ Q_2 \hat{+} Q_3) \circ (-P_1) \hat{+} I\} \tag{43}$$

In Fig. 4 the structures corresponding to eqs. (42) and (43) are compared to the minimal third-order inverse. We see that the minimal scheme (c) involves one third-order Volterra operator of Q and four of the second-order, while the non-minimal schemes (a) and (b) involve one third-order and just two second-order operators. Therefore the recursive method begins to show its effectiveness with structures resulting from at least two applications of the recursive equations. On the other hand, the complexity reduction begins to be considerable after three recursive order increments. In this case, several schemes are possible, depending on which recursive equation has been used at each step. Such schemes are all characterized by the same structural complexity. For example, applying eq. (33) twice we get the fourth-order inverse of Fig. 5. The application of Schetzen's method for the synthesis of the fourth-order minimal inverse gives rise to a structure whose complexity is enor-

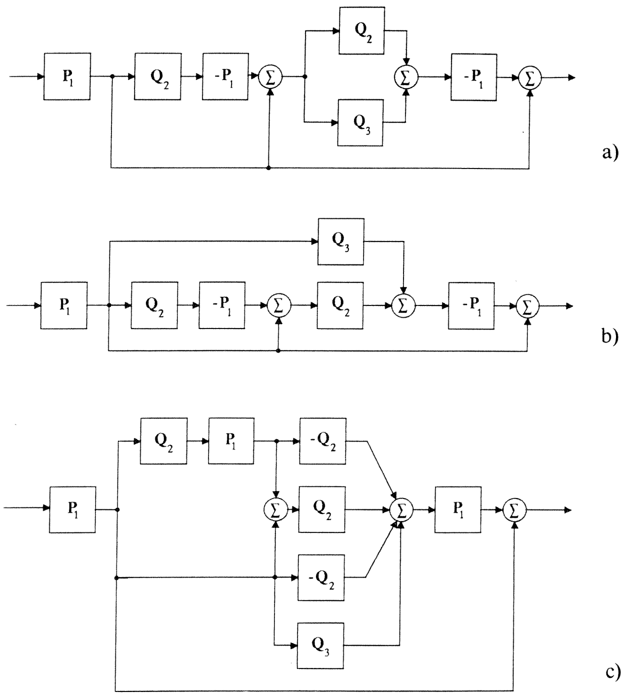

Fig. 4 - Third-order inverse: a) first recursive scheme; b) second recursive scheme; c) minimal scheme.

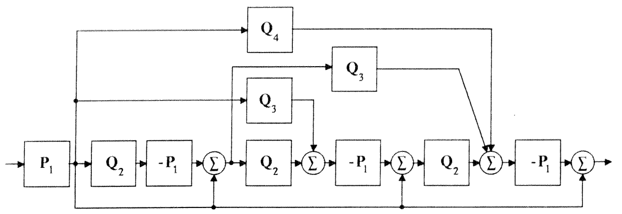

Fig. 5 - Fourth-order inverse obtained with the second synthesis scheme.

mously greater. In Fig. 6 is reported only that part of the minimal scheme corresponding to the fourth-order Volterra operator in which $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ are second- and third-order operators of the minimal inverses, respectively. The complexity of $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ is of the same ord-

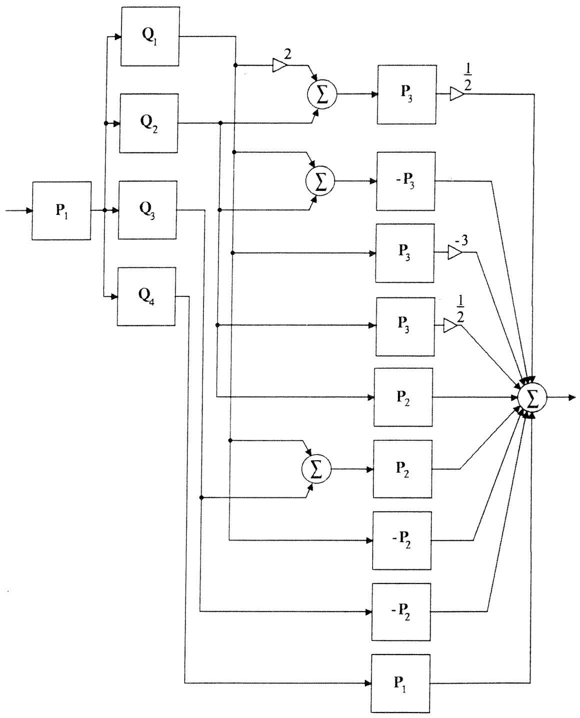

Fig. 6 - Fourth-order operator of the minimal inverse.

er as that of the systems shown in Fig. 3 and 4 c). This should give an idea of the resulting complexity. The reduction of complexity obtained with the recursive procedure is related to the fact that our method synthesizes the whole structure of the inverse while, in order to obtain a minimal inverse it is necessary to synthesize the individual Volterra operators and, consequently, to implement individual multilinear operators.

5. EXAMPLE OF APPLICATION

In digital radio transmission systems, the necessity of increasing the spectrum efficiency using high capacity modulation formats such as multilevel quadrature amplitude modulation (QAM) contrasts with the need for good power efficiency, as the former requires the entire system to work in linear regime, while the latter allows the high-power amplifier (HPA) to work near saturation, where the nonlinear distortion, acting on a band-limited pulse stream, gives rise to nonlinear intersymbol interference (ISI).

The end-to-end link could be modeled as a Volterra system [14], and we limit ourselves to consider the techniques for reducing the ISI based on the nonlinear filtering of the data stream entering the system (predistortion) or on the nonlinear equalization at the receiver. In other words, we are dealing with problems of p^{th}-order preinversion and postinversion, respectively. Such problems could be solved with the recursive method of Section 4. Others compensation methods have been proposed in the literature, as for instance the global compensation of the nonlinearity [18], but they requires a different synthesis approach than the one here reported.

5.A. *System model*

A simplified baseband model of the system in question is reported in Fig. 7. In this scheme, $\{a_n\}$ is a se-

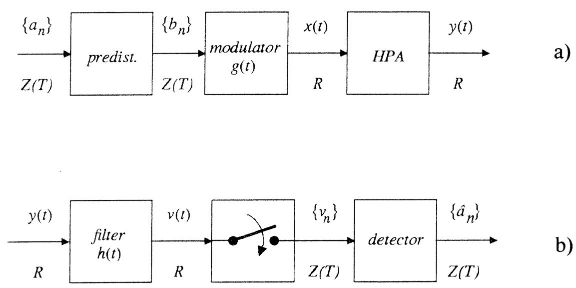

Fig. 7 - Model of the nonlinear digital transmission system: a) transmitter side; b) receiver side.

quence of T-spaced complex symbols belonging to an L-QAM constellation, therefore both $\Re e\{a_n\}$ and $\Im m\{a_n\}$ take on values $\pm 1, \pm 3, \ldots, \pm\sqrt{L}-1$. Such a sequence drives the compensator that we want to synthesize, whose output is a sequence of complex samples $\{b_n\}$. $g(t)$ is the bandlimited pulse shape of the modulator, while all the transformations between the HPA and the detector are represented by a single linear filter whose impulse response $h(t)$ satisfies the Nyquist criterion for the cancelation of the linear ISI. The HPA is modeled by a complex function over a complex domain [15] which, in general, can be described by a power series

$$y = \sum_{i=1}^{\infty} \alpha_{2i+1} |x|^{2i} x \tag{44}$$

The carrier reference and the timing are assumed to be ideal and the thresholds of the detector are set at $0, \pm 2, \pm 4, \ldots, \pm\sqrt{L}-2$. The discrete signal entering the detector is $v_n + \eta_n$, v_n being the sequence of samples at the output of the receiver and $\{\eta_n\}$ a sequence of complex samples of noise.

It is easy to verify that the system of Fig. 7 can be represented by the discrete-time Volterra series

$$v_n = \sum_{i=0}^{\infty} \sum_{k_1 \ldots k_{2i+1}} b^*_{k_1} \ldots b^*_{k_i} b_{k_{i+1}} \ldots b_{k_{2i+1}} q_{2i+1}(nT-k_1T, \ldots, nT-k_{2i+1}T) \tag{45}$$

whose Volterra kernels are

$$q_{2i+1}(n_1 T, \ldots, n_{2i+1} T) = \alpha_{2i+1} \int_{-\infty}^{\infty} h(\tau) \prod_{j=1}^{i} g^*(n_j T - \tau) \prod_{k=i+1}^{2i+1} g(n_k T - \tau)\, d\tau \quad (46)$$

The Volterra representation (45) differs from that of eq. (6) in the fact that some of the input samples are complex conjugates. The results obtained in Section 4, however, continue to hold true.

5.B. *Preinversion schemes (predistortion)*

Using the recursive eq. (33), it is easy to derive the schemes of a third-order and a fifth-order preinverse, reported in Fig. 8. In such schemes, the linear part is

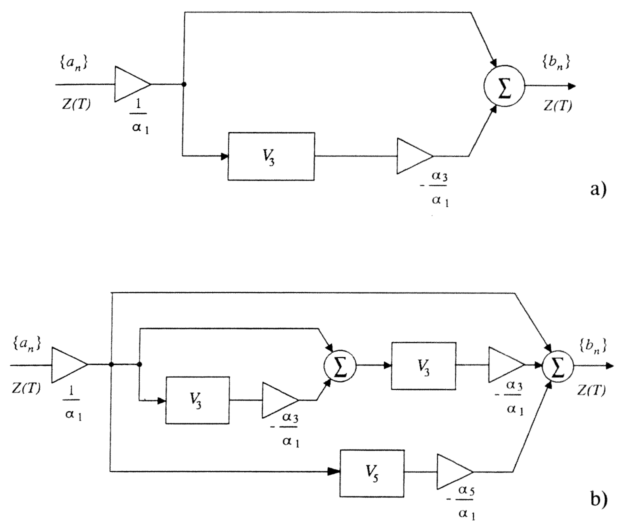

Fig. 8 - Inverses of the nonlinear digital transmission system: a) third-order predistorter; b) fifth-order predistorter.

given by a simple multiplication by $1/\alpha_1$, since the linear part of the system is a Nyquist channel. The Volterra operators V_3 and V_5 can be implemented as shown in the scheme of Fig. 9, for $k = 1$ and $k = 2$, respectively. The scheme of Fig. 9 requires to work ideally on the

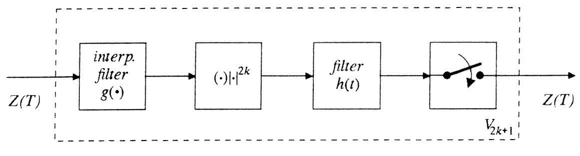

Fig. 9 - Implementation of the operator V_{2k+1} of the predistorters.

continuous time domain $\Re$ within the dashed box. However, we can work on discrete time domain $Z(T_c)$ in place of $\Re$, without loosing any information, after the following considerations: i) the signal driving the dashed block is bandlimited to $B = (1 + \gamma)/2T$, with $0 \leq \gamma \leq 1$ representing the roll-off factor, ii) the wider bandwidth of the signals involved is after the nonlinearity and it is equal to $B_k = (2k + 1)B$. Then, if we use a sampling period $T_c \leq 1/2B_k$, we guarantee a perfect signal reconstruction before sampling. If we choose $T_c = T/M$, M integer, we can build the structure of Fig. 9 with a polyphase filter with M parallel branches, each composed by a FIR working at a symbol rate $1/T$ followed by the nonlinearity and another FIR working at a symbol rate $1/T$.

We note that for a standard value $\gamma = 0.5$, we get $M = 5$ for $k = 1$, $M = 8$ for $k = 2$ and $M = 11$ for $k = 3$.

However, for the application here reported we found a residual error irrilevant for $M \geq 3$ and for the FIR filter limited to 7 taps each.

5.C. *Performance evaluation*

In order to test effectiveness of the preinversion schemes, we evaluated the performance of the system by means of a simulation program. The performance was measured in terms of the relative flat fade margin $F/F_{\max}$, F being the signal reduction that can be sustained on the link before the bit error rate (BER) reaches a given performance threshold BER_0, and $F_{\max}$ being the maximum theoretical value of F, which is a characteristic of the operative conditions. The flat fade margin was measured as a function of the HPA input back-off, which is defined as the reduction of the peak input power from its maximum value. The difference between the peak values of $F/F_{\max}$ gives the gain due to the compensators.

As shown in Fig. 10, the use of a fifth-order com-

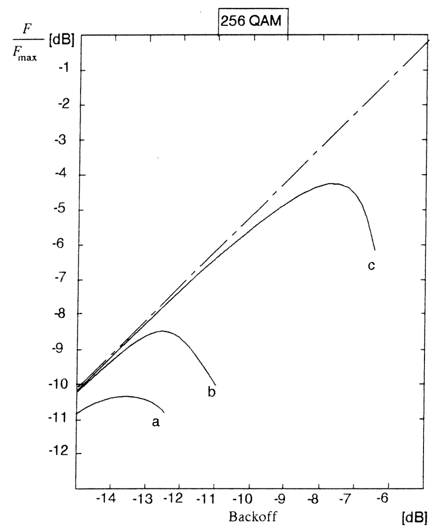

Fig. 10 - Relative flat-fade margins of a 256-QAM system: a) with no predistortion; b) with a third-order predistorter; c) with a fifth-order predistorter.

pensator [16, 17] gives rise to a gain of about 6 dB. Notice that a feasible fifth-order scheme can only be obtained with the recursive synthesis method here presented. In fact, a minimal inverse, obtained with Schetzen's technique, is feasible only in the third-order case, for which the achievable gain is limited to only 2 dB.

5.D. *Postinversion schemes (nonlinear equalization)*

As stated by Theorem 1, a p^{th}-order preinverse is also a p^{th}-order postinverse, therefore the predistorters of Subsection 5.B. can be used as nonlinear equalizers.

The compensation of channel nonlinearities in digital radio communication systems by means of a nonlinear equalizer has already been studied by Biglieri et.al.

[18]. The system described in [18], however, requires to estimate a number of parameters related to the kernels of the system to be linearized. Using the schemes of postinversion described in Subsection 5.B the only parameters to be estimated are the coefficients α_1, α_3 and α_5 of the I/O relationship (44) of the HPA, since the pulse shape is known and, with no loss of generality, the linear part of the link is assumed to be a Nyquist channel.

Notice that the recursive method allows us to build-up postinverses of order 7 or more without any calculation, while conventional techniques based on functional representations of the nonlinear system would require us to follow a long and tedious synthesis procedure.

6. CONCLUSION

A new recursive technique for the synthesis of the p^{th}-order inverse of a Volterra system, based on the operational point of view, has been presented. In the proposed procedure, we relaxed the condition that the inverse be of order p, allowing it to have residual kernels of higher order. The choice of the residuals was made with the aim of reducing the complexity of the resulting synthesis schemes and with the purpose of simplifying the synthesis method.

The recursive technique presented in Section 4, instead of synthesizing one operator at a time, yields the synthesis of the whole structure of the compensator in such a way that it is not necessary to implement any multilinear operator by means of complex interconnection of Volterra operators of the system to be inverted. Moreover, as the technique is based on simple recursive schemes, the synthesis results enormously simplified, and does not require long and tedious operations.

A comparison between minimal structures obtained by means of Schetzen's method and those resulting from the application of our recursive technique has been presented, showing that the structures that we obtain are much less complex, and that their complexity increases much more slowly as the order of inversion increases.

As an example of application of the proposed method, we have considered the linearization of a digital radio link in which the high-power amplifier is operated near saturation. Two preinversion schemes have been presented and the compensated system has shown a considerable performance improvement.

A research aspect to be investigated, concerning the synthesis approach which has been presented in this paper, is the possibility of making an appropriate choice of the residual operators in order to optimize other system characteristics.

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