

Joint sublattice selection and prefilter design for the optimal decimation of 2-D digital signals

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Abstract: Signal decimation aimed at optimal spectral packing has a variety of applications in areas ranging from array processing to image processing. The authors propose and discuss a new method for determining the decimation grid and prefilter that best fit the spectral extension of any 2-D signal defined on an arbitrary sampling lattice. The method first quantifies the spectral anisotropy through the determination of the principal axes of the power spectrum, then it selects among all possible decimation grids those that are compatible with the spectral extension shaped on the 'inertia' ellipse. Finally, for each of them it geometrically constructs the ideal prefilter whose convex passband best encircles this spectral extension. A final selection is thus made among the available sublattice/prefilter pairs according to some specific criterion. The method, implemented in a fully automatic computer procedure, has been tested over several digital images to evaluate its performance in terms of the impact of the spectral truncation on the overall quality of the reconstructed images.

1 Introduction

Sampling multidimensional analogue signals causes their spectrum to replicate over a regular point structure whose density is inversely proportional to the sampling density. The 'degree of sparseness' of the spectral replicas can be thought of as a measure of the 'inefficiency' of the sampling process (*spectral redundancy*). Minimising the gap among spectral replicas is known to reduce information redundancy [1], which would definitely be useful in a variety of applications that range from array processing [2, 3] (radar sonar and seismic) to image processing [1, 4-6]. Spectral packing through decimation is not an easy task as it consists not just of a rational selection of data samples, but it also needs a careful spectral truncation for avoiding aliasing. To perform antialiasing prefiltering, knowing the area of the spectral extension (*spectral*

occupancy), is not sufficient: we also need to consider its shape.

To be able to reduce spectral redundancy through decimation, the first problem encountered is that of generating all possible sublattices of an assigned *order* (density reduction ratio). Given a sampling lattice, the number of possible distinct decimation grids of a given order is finite, although it increases very rapidly with the order. It should thus be clear that, to be able to make an appropriate selection of the subgrid according to the spectral properties of the signal, not only do we need to define and estimate the spectral extension of the signal but also we need some criteria for limiting the search space of candidate sublattices by ruling out those that do not meet some specific conditions on the spectral extension.

Another nontrivial problem is that of deciding upon a definition of spectral extension that could be fruitfully used, not just for the selection of the decimation grid, but also for the design of the antialiasing filter. Actually, the spectral energy of nonsynthetic 2-D signals usually occupies regions with a quite irregular and complex shape [6], which makes the optimal design of a prefilter very difficult. It is important to note, however, that the spectral extension may be defined and estimated according to the class of prefilters adopted for decimation purposes. More specifically, if the class of prefilters in which we are interested is restricted to those that have convex passband we do not need a detailed description of the spectral extension shape. In fact only the direction around which the spectral energy is maximally concentrated and a measure of the energy dispersion about it needs to be determined. This way of quantifying the anisotropy of the spectral distribution corresponds to approximating the spectral extension with an ellipse whose shape is decided by the ratio between the inertia moments of the power spectrum, while its size is chosen according to the severity of the spectral truncation that we are willing to apply.

Restricting the class of prefilters to those having a convex passband region, that is, adopting a second-order model for the spectral extension, seems quite a reasonable choice since it allows us to detect the dominant direction over which high frequency portions of the 2-D signal are most concentrated, and to adopt decimation grids that exhibit a higher density of samples in that specific direction. According to our experience, any further extension of the class of prefilters used would only modestly improve the quality of the signal reconstructed after decimation, and would not justify the heavy complications that would arise from it.

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A method for jointly and automatically determining a decimation grid and prefilter for two-dimensional discrete signals defined on arbitrary lattices is proposed and tested. The first step consists of estimating the principal axes of the power spectrum. From these the maximum decimation order is computed as the ratio k_{MAX} between the area of one period of the spectrum of the original signal and that of the spectral ellipse (i.e. an ellipse whose axes are proportional to the inertia axes). Among all possible decimation grids of order $k \leq k_{MAX}$ (which can be automatically catalogued using a Hermite-basis representation) we select those that are *compatible* with the elliptical spectral extension. In other words, only the decimation grids that do not give rise to aliasing between *elliptic* truncations of the spectral extension of the signal are kept. If there are no compatible decimation grids of a certain order, then the search among grids of lower order must be repeated until some compatible grids are found. For each compatible decimation grid an ideal prefilter is generated by following a geometric approach, and the best decimation grid is selected according to some specific criterion. For example, we may choose the sublattice which is associated to the prefilter that has the most uniformly distributed gap between the ellipse and the border of the passband region.

The technique proposed in this article has been implemented in a fully automatic computer procedure and applied to a variety of 2-D test signals. To make the examples of application as intuitive and visual as possible, the signals used are nonsynthetic images, each having different spectral properties. It is important to emphasise that although in the examples of application considered the signals originally are defined on rectangular grids, the proposed decimation technique works equally well on signals defined on nonseparable lattices, as no specific hypotheses on their structure are required.

The following Section includes all the basic information that is needed for the comprehension of the concepts discussed thereafter. In particular, a brief summary of the properties of sampling lattices and sublattices is included together with a review of concepts of Fourier analysis on lattice structures. Readers who are already familiar with the theory of sampling lattices may ignore this Section, while those who would like a more complete introduction to lattice structures, may refer to [2, 4, 5, 7, 8]. To facilitate reading, a glossary of terms is provided in the Appendix (Section 7.1).

2 Mathematical preliminaries

To approach the decimation problem in a general way, we need a class of regular point sets with sufficient algebraic structure to be able to define the Fourier transform and, thus, to perform spectral analysis. Such sampling structures are called *lattices* [2, 4, 7, 8], and are characterised by the property of being invariant with respect to the translation [7].

2.1 Lattices and sublattices

The M -dimensional lattice Λ generated by a nonsingular matrix $\mathbf{A} \in R^M$ is defined as

$$\Lambda = \text{LAT}(\mathbf{A}) = \{\mathbf{x} \in R^M | \mathbf{x} = \mathbf{A}\mathbf{n}, \mathbf{n} \in Z^M\}$$

which is the set of all possible linear combinations, with integer coefficients of the M linearly independent vectors (basis of the lattice) represented by the columns

of \mathbf{A} . Each basis generates a unique lattice, while a lattice may have several bases.

Given a basis \mathbf{A} , it is possible to derive all other bases of the lattice $\Lambda = \text{LAT}(\mathbf{A})$, by using the fact that the generic basis \mathbf{A}' can always be written as $\mathbf{A}' = \mathbf{A}\mathbf{U}$, where $\mathbf{U} \in Z_M$ is a *unimodular* matrix, that is, an integer matrix whose determinant has absolute value equal to one (see Appendix, Section 7.2). As a consequence, all bases of a lattice must have the same *modulus* (absolute value of its determinant). The modulus $d(\Lambda)$ of a lattice Λ has a meaningful geometric interpretation, which is related to the definition of its *fundamental cell*.

A fundamental cell S of an M -dimensional lattice Λ is a (nonnecessarily connected) closed region of R^M such that the collection $S_\Lambda = \{S + \mathbf{a}, \mathbf{a} \in \Lambda\}$ of all shifted version of S on all points of the lattice *tiles* the whole space R^M without overlappings between distinct translations. It is not difficult to verify that there exist infinite fundamental cells for a single lattice, but their hypervolume is always the same. Since the hyperparallelogram \mathcal{P} corresponding to the vectors of the basis \mathbf{A} is a fundamental cell of the lattice $\Lambda = \text{LAT}(\mathbf{A})$ and its volume is equal to $d(\Lambda)$, we conclude that $d(\Lambda)$ is the volume of all fundamental cells of Λ , therefore $1/d(\Lambda)$ can be interpreted as a measure of the lattice *density*. The fundamental cell is very important in problems of multidimensional sampling and interpolation, and some of its properties will be extensively used in the following Section. Unfortunately, there exist no general geometric classifications of all possible fundamental cells of a given lattice. The only results that are available in the literature concern convex cells [8] and are particularly simple in the two-dimensional case, in which case the only convex regions that tile R^2 are parallelograms and hexagons with central symmetry.

The concept of *sublattice* is particularly important for the decimation problem. The decimation grids considered here are, in fact, sublattices of the original sampling grid Λ , that is, subsets of Λ that have a lattice structure. To be able to select a suitable sampling grid for a given digital signal, it is of paramount importance to be able to classify and generate all possible sublattices of a given order, that is, those that have a preassigned decimation ratio.

Given an M -dimensional lattice $\Lambda = \text{LAT}(\mathbf{A})$, the lattice $\Gamma = \text{LAT}(\mathbf{B})$ is a sublattice of Λ if and only if there exists a nonsingular integer matrix \mathbf{H} such that $\mathbf{B} = \mathbf{A}\mathbf{H}$. This result [9] is particularly important as it provides us with a method for analytically generating all sublattice bases. The integer number $|\det(\mathbf{H})| = |\det(\mathbf{B})|/|\det(\mathbf{A})|$ represents the ratio between the densities of Λ and Γ , and is often referred to as the *index* of Γ in Λ . This number, which is also written as $(\Lambda : \Gamma)$, corresponds to what we have already called the *decimation ratio* from Λ to Γ . Note, however, that a lattice may have several bases, therefore the above result is not enough for classification purposes. The problem of automatically generating all k th order sublattices of a given M -dimensional lattice $\Lambda = \text{LAT}(\mathbf{A})$, however, has already been solved [6] and corresponds to finding a partition of the set $\mathcal{W} = \{\mathbf{W} \in R_M : \mathbf{W} = \mathbf{A}\mathbf{K}, \mathbf{K} \in Z_{M,k}\}$ into distinct classes \mathcal{W}_i each of which identifies a unique sublattice. To do so, we can look for a partition of the set $Z_{M,k}$ into distinct classes \mathcal{K}_i of right-equivalent matrices. More specifically, we would like each class \mathcal{K}_i to univocally determine a sublattice $\Gamma_i = \text{LAT}(\mathbf{W}_i)$ of Λ through the relationship $\mathbf{W}_i = \mathbf{A}\mathbf{K}, \mathbf{K} \in \mathcal{K}_i$. According

to the Hermite normal form theorem (see Appendix, Section 7.2) each class of right-equivalent matrices of $Z_{M,n}$ contains one and only one matrix in Hermite normal form. Consequently, all k th order sublattices of $\Lambda = \text{LAT}(\mathbf{A})$ will be given by $\Gamma_i = \text{LAT}(\mathbf{W}_i)$, where $\mathbf{W}_i = \mathbf{A}\mathbf{H}_i$, $\mathbf{H}_i \in \mathcal{H}_{M,k}$, where $\mathcal{H}_{M,k}$ is the set of integer matrices in Hermite normal form whose determinant is equal to k .

The number of distinct k th order sublattices of a given lattice Λ corresponds to the number of distinct matrices in Hermite normal form with determinant k , that is, on the number of possible integer factorisations of k . Such a number increases very rapidly with the decimation index k (and the dimension of Λ). For example, there exist 91 distinct sixth-order sublattices of a 3-D lattice while, in the 2-D case, the number drops down to 12.

2.2 Fourier transform

The Fourier transform pair is defined on a lattice $\Lambda = \text{LAT}(\mathbf{A})$ as

$$\begin{aligned} U(\mathbf{f}) &= \sum_{\mathbf{x} \in \Lambda} u(\mathbf{x}) e^{-j2\pi\mathbf{f}^T \mathbf{x}} \\ &= \sum_{\mathbf{n} \in Z^M} u(\mathbf{A}\mathbf{n}) e^{-j2\pi\mathbf{f}^T \mathbf{A}\mathbf{n}}, \quad \mathbf{f} \in R^M \\ u(\mathbf{A}\mathbf{n}) &= |\det(\mathbf{A})| \int_{\mathcal{P}} U(\mathbf{f}) e^{j2\pi\mathbf{f}^T \mathbf{A}\mathbf{n}} d\mathbf{f}, \quad \mathbf{n} \in Z^M \end{aligned} \quad (1)$$

The Fourier transform $U(\mathbf{f})$ is periodic and its periodicity centres are given by the reciprocal lattice [4, 9]) $\Lambda^* = \text{LAT}(\mathbf{A}^{-T}) = \{\mathbf{y} \in R^M: \mathbf{y}^T \mathbf{x} \in Z, \mathbf{x} \in \Lambda\}$. The Fourier transform $U(\mathbf{f})$ is thus completely specified by its values in any fundamental cell \mathcal{P} of Λ^* .

2.3 Decimation and interpolation

In the multidimensional case several different subgrids having the same decimation ratio are available. This fact is a source of complications with respect to the one-dimensional case, but it also offers greater freedom in the decimation setup. Additional complications are caused by the fact that for a single decimation grid there exists a nonnumerable multitude of antialiasing filters, which can always be chosen in such a way to favour some spectral regions instead of others. To understand the reason of this increased freedom, we need operators that will be extensively used in the following Sections: the ideal interpolator and the decimator.

The interpolation can be thought of as the cascade of an elementary interpolator (zero interleaving) and a filter. More specifically, the elementary interpolation of a signal $s(\mathbf{x})$, from $\Gamma_1 = \text{LAT}(\mathbf{A}_1)$ to $\Gamma_2 = \text{LAT}(\mathbf{A}_2)$, Γ_1 being a sublattice of Γ_2 , is defined as $w(\mathbf{x}) = s(\mathbf{x})$ for $\mathbf{x} \in \Gamma_1$, and $w(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma_2$ but $\mathbf{x} \notin \Gamma_1$. It is not difficult to verify [4] that $W(\mathbf{f}) = S(\mathbf{f})$, which means that the ideal interpolation does not affect the Fourier transform but just the periodicity lattice which is now Γ_2^* .

Decimating u from $\Gamma_1 = \text{LAT}(\mathbf{A}_1)$ to $\Gamma_2 = \text{LAT}(\mathbf{A}_2)$, where Γ_2 is now a sublattice of Γ_1 (i.e. $\Gamma_2 \subset \Gamma_1$, and $\mathbf{A}_1^{-1} \mathbf{A}_2 = \mathbf{M}$ unimodular) returns the signal $v(\mathbf{x}) = u(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$. In this case the relationship between Fourier transforms [4] turns out to be

$$V(\mathbf{f}) = \frac{1}{(\Gamma_1 : \Gamma_2)} \sum_{\mathbf{a} \in \mathcal{I}} U(\mathbf{f} + \mathbf{a}) \quad (2)$$

\mathcal{I} being any Γ_1^* -period of Γ_2^* ; and $(\Gamma_1 : \Gamma_2) = |\det(\mathbf{M})|$ the decimation factor.

To be able to perfectly reconstruct a signal u defined on Γ_1 from its decimated version v on Γ_2 , that is, for the decimation to be invertible, it is necessary for the support of $U(\mathbf{f})$ to be confined inside some fundamental cell \mathcal{P}_2 of Γ_2^* . In this case the reconstruction can be done by using the cascade of an ideal interpolator from Γ_2 to Γ_1 and an ideal filter with frequency response

$$H(\mathbf{f}) = \begin{cases} (\Gamma_1 : \Gamma_2) & , \mathbf{f} \in \mathcal{P}_2 \\ 0 & , \mathbf{f} \in \mathcal{P}_1, \mathbf{f} \notin \mathcal{P}_2 \end{cases} \quad (3)$$

As the impulse response of the filter is defined on Γ_1 , the frequency response $H(\mathbf{f})$ will be a Γ_1^* -periodic function of R^2 ; therefore, it is sufficient to specify it in a Γ_1^* -period \mathcal{P}_2 of R^2 .

3 Decimation approach

According to the results summarised in Section 2, decimating a signal on a k th order sublattice causes its spectrum to replicate $k - 1$ times according to a geometry which is completely specified by the reciprocal lattice of the decimation grid. Section 2 also showed what the requirements for a region of the frequency plane to become a fundamental cell of a specific lattice are, and how to use this information for designing an ideal prefilter. What we need now is a strategy for approaching signal decimation problem in its globality. More specifically, as we are interested in reducing spectral redundancy, not only do we need a systematic strategy for modelling the spectral extension of a generic two-dimensional digital signal, but also we need to be able to interpret this information to make an optimal selection of the decimation grid and, at the same time, to design the antialiasing filter that best fits the spectral extension.

It is important to note that sublattice selection and prefilter design cannot be considered as two separate and independent steps. In fact, the shape of a fundamental cell to be used for designing the prefilter depends on the lattice structure, while the choice of lattice depends on the shape of cell with which to encircle the spectral extension. Designing prefilter and sublattice jointly, however, is not a simple problem, as the two operations cannot be done simultaneously. The method proposed here solves the problem by using all the information held about the spectral extension for drastically reducing the number of candidate subgrids to be considered and geometrically constructing a prefilter for each one of them. A final selection is then made according to some specific criterion.

The scheme of Fig. 1 summarises the approach we propose for decimating two-dimensional signals. The first step consists of estimating the second-order spectral extension of the signal through the evaluation of the inertia axes of its power spectrum. We can then determine an upper bound for the index of the sublattices to choose among from the spectral occupancy of the estimated spectral ellipse. The maximum decimation index k_0 is the order from which to start looking for suitable decimation grids.

Given a decimation order k , all distinct k th order sublattices are generated and, among them, all those that are *noncompatible* with the spectral extension, that is, those that cause spectral ellipses to overlap, are ruled out. If no compatible candidate subgrids can be

found then the search will be repeated among sublattices of an inferior order. For each compatible sublattice, we geometrically generate the fundamental cell that best fits the elliptical spectral extension. A selection of the best sublattice-cell pair can finally be performed among the remaining candidates according to some specific criterion.

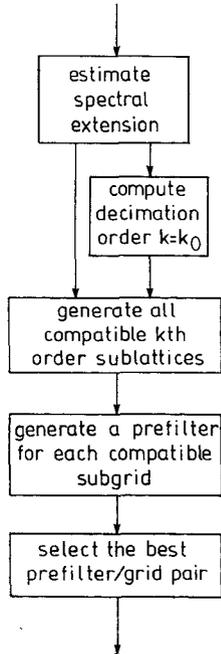


Fig. 1 Global approach to the decimation problem

This Section addresses all the above-mentioned problems, proposing a solution for each one of them. The proposed method can be organised in an automatic computer procedure.

3.1 Spectral extension estimation

We have already seen in Section 2.3 that decimating a signal defined on a lattice causes replication of the already periodic spectrum of the signal. The number of replicas is equal to the decimation ratio and the geometry of the resulting replicas depends on the decimation grid. It should be quite clear that to decimate the signal with a minimum loss of information the spectral extension of the signal to be decimated must be determined. In other words, we should estimate a spectral region that contains the most significant information on the original signal. Such a region will be used later on for the selection of a decimation grid and an antialiasing filter that do not cause overlapping between spectral replicas.

In this Section a definition of spectral extension of a 2-D digital signal is given and a parametric model that simplifies its estimation is proposed. Then, we show how to estimate the parameters of the spectral extension model from the analysis of the 2-D signal. Finally, we show how to determine an upper limit to the decimation ratio from the analysis of the estimated spectral extension.

3.1.1 Spectral extension model: The spectral extension of a 2-D discrete signal $u(\mathbf{x})$, $\mathbf{x} \in \Lambda$, can be simply defined as

$$\mathcal{E}_u = \{\mathbf{f} \in \mathcal{P} : |U(\mathbf{f})| > S > 0\} \quad (4)$$

that is, the portion of one Λ^* -period \mathcal{P} of R^2 where the

magnitude of the spectrum $U(\mathbf{f})$ of $u(\mathbf{x})$ exceeds an assigned threshold S . The *bandwidth*, or *spectral occupancy* of u can thus be defined as $B = \text{area}(\mathcal{E})$, and its normalised value $e = \text{area}(\mathcal{E})/\text{area}(\mathcal{P})$, $0 < e < 1$, can be used as an index of the spectral efficiency.

When dealing with nonsynthetic signals such as 2-D images, the definition (eqn. 4) of spectral extension usually produces sets \mathcal{E} that are quite complex in shape. In fact, irregular or even scattered regions [6] are usually obtained. In practice, it is extremely difficult to determine decimation grids and relative fundamental cells that best encircle spectral extensions such as those obtained through eqn. 4. The spectral energy of nonsynthetic 2-D signals, however, is normally concentrated near the origin and its distribution is often limited to a connected region of the frequency plane. These two facts suggest that it would be wise to make some simplifying assumptions on the structure of the prefilter that we wish to design. It is thus reasonable to restrict the class of prefilters in which we are interested to those that have a *compact* and *convex* passband region. In fact, as explained in Section 2.1, the only convex regions that tile R^2 are parallelograms and hexagons with central symmetry. Therefore, we no longer need information on the spectral extension shape which is as detailed as that provided by eqn. 4. We simply need to determine the direction around which the spectral energy is maximally concentrated (principal axis) and to measure of the energy dispersion about that axis. In other words a second-order model of the spectral energy distribution is sufficient. Quantifying the anisotropy of the spectral distribution through a second-order model corresponds to approximating the spectral extension with an ellipse whose shape is decided by the inertia moments of the power spectrum while its size is chosen according to the severity of the spectral truncation that we are willing to apply.

3.1.2 Estimation of the spectral extension: Any nonsynthetic discrete 2-D signal $u(\mathbf{x})$, $\mathbf{x} \in \Lambda$, has a limited region of support $Q \subset R^2$, therefore we may consider its Ψ -periodicised version $u_\Psi(\mathbf{x})$, Ψ being some sublattice of Λ , so that the spectrum $U_\Psi(\mathbf{f})$ of $u_\Psi(\mathbf{x})$ results as being discrete over Ψ^* and periodical over Λ^* . It should be quite clear that, if the Ψ -periodicisation is correct, then the frequency sampling associated to it does not cause any loss of information. It is thus possible to derive all spectral information about the signal $u(\mathbf{x})$ through DFT computation.

To estimate the second-order spectral extension associated to the power spectrum $S(\mathbf{f}) = |U(\mathbf{f})|^2$, the energy distribution about its symmetry centre is computed the same way we would proceed with computing the mass distribution of a body. More specifically, we can think of the power spectrum samples as masses characterised by a certain location on the frequency plane. The second-order model provides an indication of how the energy is distributed on the frequency plane. Such a model is based on the determination of the principal inertia axes d_1 and d_2 and the relative radii of gyration ρ_1 and ρ_2 of the power spectral samples. The axes d_1 and d_2 are determined from the eigenvectors of the inertia matrix (matrix of the inertia moments), while the radii $\rho_1 = \sqrt{I_1/M}$ and $\rho_2 = \sqrt{I_2/M}$ depend on the principal inertia moments I_1 and I_2 (eigenvalues of the inertia matrix) and the total energy M of the 2-D signal.

With respect to the reference frame (d_1, d_2) , the inertia ellipse is described by the equation $\rho_1^2 d_1^2 + \rho_2^2 d_2^2 = 1$ therefore the second-order model of the spectral extension is a set of the form

$$\rho_1^2 d_1^2 + \rho_2^2 d_2^2 \leq r^2 \quad (5)$$

where r is a scaling factor for the radii of gyration, to be chosen in order for the spectral ellipse (eqn. 5) to contain the most significant portion of the power spectrum (e.g. by selecting a threshold for the signal energy included in the ellipse). The spectral occupancy of the signal, that is, the area $A = \pi\rho_1\rho_2/r^2$ of the spectral ellipse, provides us with an upper bound for the index k of the sublattices that could be used for decimating the signal. In fact, we must have $k \leq k_0$, where k_0 is the largest integer which is not greater than $\text{area}(\mathcal{P})/A$ and \mathcal{P} is any fundamental cell the reciprocal sampling lattice.

3.2 Design of prefilter and decimation grid

Now that the maximum decimation ratio k_0 is available, all k th order sublattices with $k \leq k_0$ are required to be determined. We have already seen in Section 2 that such a problem can be solved by generating all matrices in Hermite normal form that have determinant equal to k . In this Section we show how to use the spectral extension estimate for designing sublattice and prefilter according to the global scheme proposed at the beginning of Section 3.

3.2.1 Compatible sublattices: Given the elliptical spectral extension of a 2-D signal defined on the lattice Λ , a method for deciding whether a sublattice Γ is compatible with it, that is, whether the spectral replicas generated by the Γ -decimation of the signal overlap, is required.

The first step consists of determining the k points of the reciprocal lattice Γ^* that fall inside one Λ^* -period of R^2 . These points correspond to the centres of replication of the original spectrum. In principle, the compatibility check should test whether any two replicas of the spectral extension overlap. In practice, because of the algebraic properties of lattices, it is sufficient to check that the replica placed at the origin of the frequency plane overlaps with none of the others. This can be done by determining the limit-region for the replication centres beyond which no overlapping occurs and making sure that all points of the reciprocal lattice of the subgrid fall outside it.

Since the spectral extension model is elliptical, the compatibility check is particularly simple. Given an ellipse, the curve described by the centres of all of its tangent replicas is itself an ellipse whose radii of gyration are twice the ellipse's radii. Such a curve is also referred to as *threshold ellipse* ξ , and its equation is $(2\rho_1)^2 d_1^2 + (2\rho_2)^2 d_2^2 = r^2$.

A sublattice Γ is compatible with the spectral extension of a 2-D digital signal if none of the replicas overlap with the original extension, that is, if all points of Γ^* , lie outside the threshold ellipse. Letting (a, b) be the coordinates of the generic point of Γ^* , referred to the principal axes of the ellipse (see Fig. 2), the compatibility check becomes

$$(2\rho_1)^2 a^2 + (2\rho_2)^2 b^2 > r^2 \quad (6)$$

The compatibility check explained above allows us to determine a restricted set of candidates for the decimation as follows:

- (i) Let $k = k_0$.
- (ii) Find all k th order sublattices of Λ and their reciprocal lattices as follows:
 - a find all distinct matrices $\mathbf{H}_i \in \mathcal{H}_{2,k}$
 - b compute the bases of all k th order sublattices as $\mathbf{B}_i = \mathbf{A}\mathbf{H}_i$
 - c compute the reciprocal bases $\mathbf{B}_i^* = (\mathbf{B}_i)^{-T}$.
- (iii) For each basis \mathbf{B}_i^* , check the condition (eqn. 6) for the eight points that are closest to the origin.
- (iv) If there are no compatible sublattices then $k \leftarrow k - 1$ and go back to step (ii), else stop.

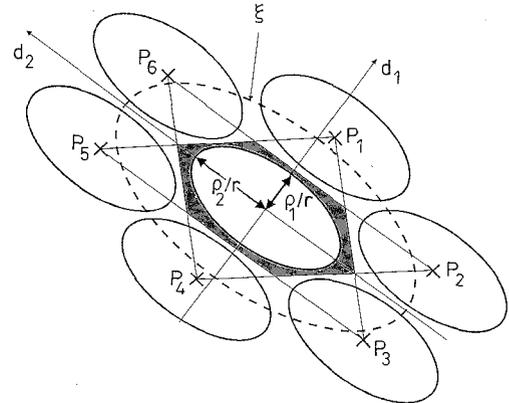


Fig. 2 Construction of the prefilter
The hexagon is given by the intersection between the two triangles $\{P_1, P_3, P_5\}$ and $\{P_2, P_4, P_6\}$ built on the six closest points of the lattice. The threshold ellipse ξ for checking the compatibility is also shown

3.2.2 Prefilter geometry: All compatible subgrids are already good candidates for decimation, although we need a criterion for deciding between them. Since the choice of decimation grid is strongly influenced by the shape of the passband region of the prefilter (fundamental cell of the reciprocal of the sublattice), to be able to decide among the compatible subgrids a fast method for generating a fundamental cell that 'well fits' the spectral extension, for each compatible sublattice, is required. More specifically, the compact and convex fundamental cells that we seek must enclose the elliptical spectral extension entirely, and the principal directions of the prefiltered spectrum must be as close as possible to those of the original spectrum.

In general, there exists a nonnumerable multitude of fundamental cells for a sublattice, and the arbitrariness of their shape is sufficient to make the search extremely difficult. Restricting the class of fundamental cells to the convex class, however, greatly simplifies the situation. According to Section 2, all convex fundamental cells of a two-dimensional lattice are hexagons with central symmetry. A method for determining a hexagonal fundamental cell of a given compatible sublattice, which entirely encircles the elliptical spectral extension, is described in Fig. 2. The method consists of determining the six points of Γ^* that lie the closest to the threshold ellipse and building two triangles by using two triplets of alternate points, as shown in Fig. 2. The desired hexagon will be the intersection of such two triangles.

The fundamental cell obtained can be used for constructing both the antialiasing and the reconstruction filters. Both will have a passband region that resembles the fundamental cell determined above. The shape of the resulting fundamental cells tends to capture the spectral anisotropy in a natural fashion as the compat-

ble sublattices are selected using all the information we have on the spectral extension.

Note that, for the above geometric procedure to be applicable, we shall assume that the threshold ellipse is entirely included in the fundamental cell of Λ^* . This limitation however, is not very restrictive for the 2-D signals of interest. The proposed decimation procedure is worthwhile applying, only to signals that are not very efficiently sampled.

3.2.3 Final selection of the decimation setup:

The final step of the decimation procedure consists of choosing the best decimation setup among the compatible subgrid/prefilter pairs. Note that all the available candidates are acceptably good, therefore the choice must be made according to some criterion of optimality that takes into account, for example, some measure of the *fitness* between fundamental cell and spectral extension. One must bear in mind, however, that the computational complexity of the prefilter should be kept reasonably modest, and this can be done by making the transition band as uniform as possible. Of course such a criterion is rather arbitrary as the spectral extension is estimated through a second-order model. Thus, we do not have very precise information about its shape. Yet, different choices of compatible sublattices often give rise to cells that are quite different from each other. Therefore, such a criterion is well-justified.

An operative way of proceeding with the application of the above criterion consists of selecting the sublattice whose prefilter has minimum impact on the principal axis of the power spectral distribution. More precisely, we can choose the prefilter that minimises the angle between the principal axis of the nonprefiltered spectrum and that of the prefiltered spectrum.

The proposed criterion, however, is not the only one possible. For example, a choice could be made according to some other criterion that takes into account computational load and memory requirements of the resulting filter implementation. For example, choosing hexagons that have one side of negligible length would simplify the implementation.



Fig.3 Original test image 1

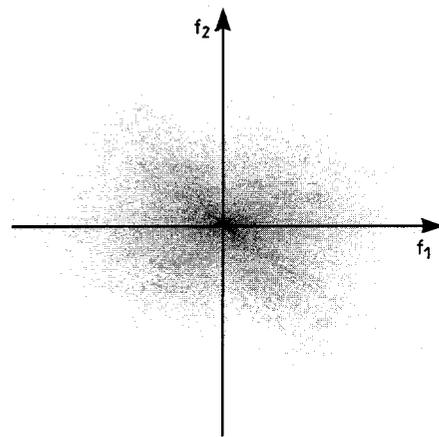


Fig.4 Power spectrum of original test image 1 (log scale)

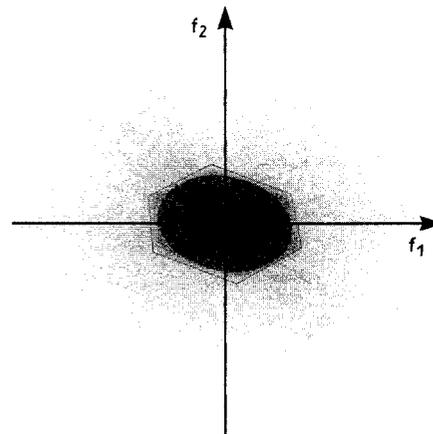


Fig.5 Estimated spectral extension and prefilter for test image 1

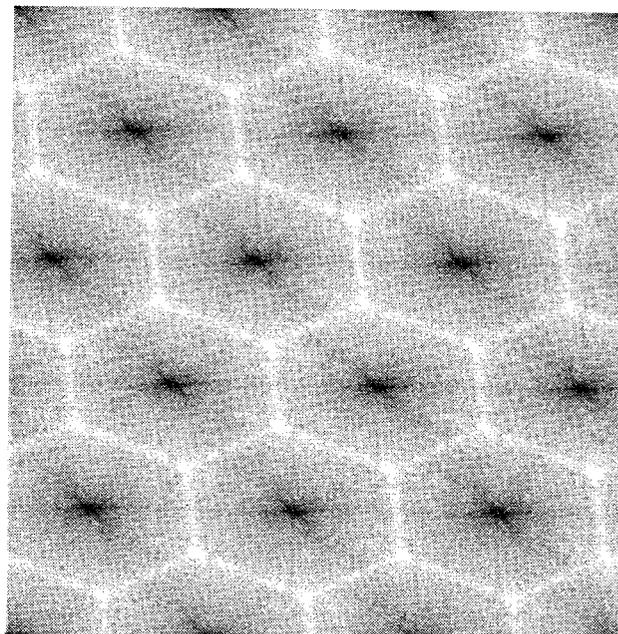


Fig.6 Spectrum of test image 1 after decimation with the 15th order decimation basis $v_1 = [5 \ 0]^T$, $v_2 = [2 \ 3]^T$

4 Examples of application

As already mentioned in Section 1, the search for a decimation grid and the generation of a prefilter according to the method proposed in this article has been implemented into a completely automatic computer procedure and tested over a series of real images. Considering that the 2-D test signals selected are images, a few remarks on the quality of the reconstruction are in order.

The ideal prefilter obtained with the proposed method is characterised by sharp transitions at the border of the passband region. Abrupt spectral truncation, however, may give rise to undesired 'ringing' effect which is highly visible as it results in artifacts that are parallel to the image edges. (It is well-known that our visual system is sensitive to noise and disturbance in a highly anisotropic way [10]. For example, a disturbance that is parallel to edges is much more visible than one that is perpendicular to it.) In principle, a prefilter should be designed taking into account the impulse response as well as the frequency response [11, 12]. In practice, for the extent of this article, it is sufficient to smooth the transitions of the hexagonal prefilter in their implementation.

A first example of application is represented by test image 1, 'Lenna', shown in Fig. 3. The spectrum of this image, reported in Fig. 4, exhibits a certain anisotropy, which is made quite visible by the elliptical spectral extension of Fig. 5, and can mainly be attributed to the prevalence of detail introduced by the brim of the hat. The principal axes of the elliptical spectral extension have been chosen to be 3.5 times the inertia axes of the power spectrum samples. The maximum order of decimation in which some compatible sublattices can be found is $k = 15$. Among the hexagonal fundamental cells (built by using the method of Section 3.2.2) that are associated to all compatible 15th order sublattices, the one whose principal axes are closest to those of the elliptical spectral extension is chosen to define the passband region of the image prefilter (see Fig. 5), while the relative sublattice, whose basis is given by

$$H = \begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix}, \quad |\det(H)| = 15$$

is the corresponding decimation grid.

At this point the image can be prefiltered through spectral windowing by using a smoothed version of the ideal prefilter obtained above. Decimating the image over the selected subgrid causes the truncated power spectrum to replicate like in Fig. 6 where it is quite apparent how the elongation of the power spectrum due to the prevalence of some edges aligned along a specific direction, causes subgrid and prefilter to preserve the spectrum in that direction.



Fig. 7 Reconstructed test image 1



a



b

Fig. 8 Reconstruction results for test image 1 with the 15th order decimation basis $v_1 = [5 \ 0]^T$, $v_2 = [2 \ 3]^T$
a Zoomed-in detail of original image
b Zoomed-in detail of reconstructed image

The same filter used for avoiding aliasing is here used for reconstructing the original image from the decimated one. The reconstruction results of test image 1 are reported in Fig. 7. A comparison between corresponding zoomed-in details of original and reconstructed images are shown in Fig. 8a and b. As we can see, the blurring due to the low-pass antialiasing filtering is still acceptable, considering the reduction of 15 times in the amount of samples that are actually being used for describing the image itself. Another reconstruction example is shown in figs. 9–12 and Figs. 13 and 14a, b. In this case the spectral extension exhibits a different orientation in the spectral anisotropy.

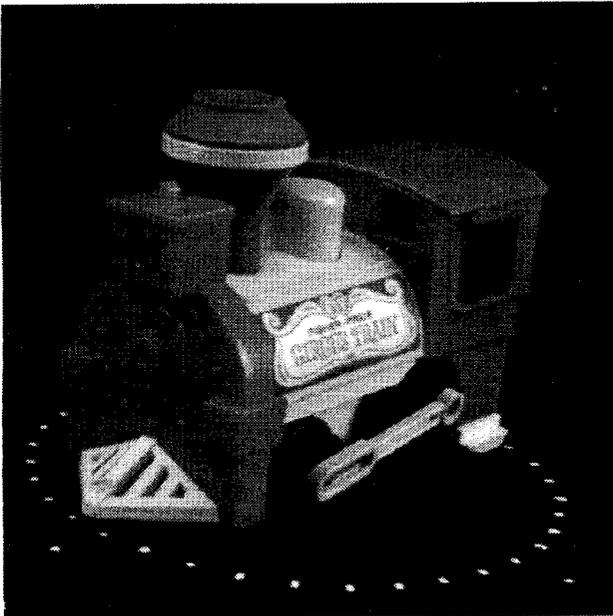


Fig. 9 Original test image 2

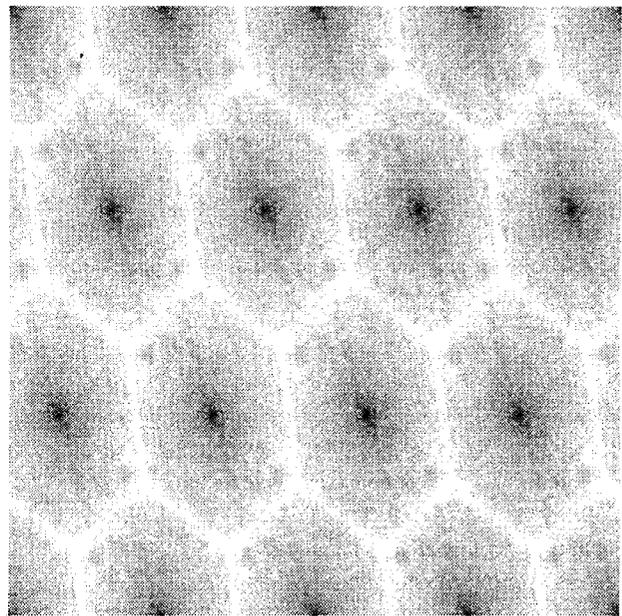


Fig. 12 Spectrum of test image 2 after decimation with the 12th order decimation basis $v_1 = [3 \ 0]^T$, $v_2 = [1 \ 4]^T$

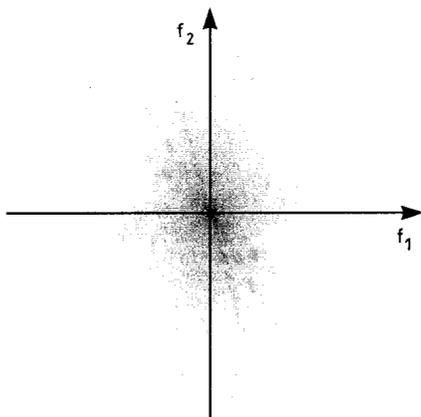


Fig. 10 Power spectrum of original test image 2 (log scale)

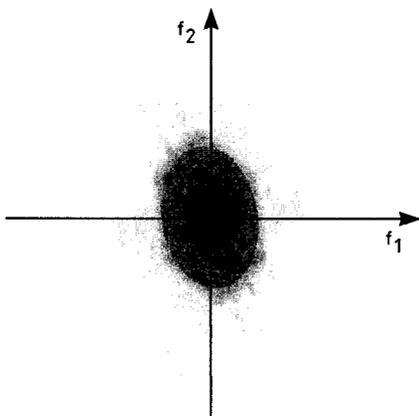


Fig. 11 Estimated spectral extension and prefilter for test image 2

5 Conclusions

This paper has presented a new technique for decimating discrete 2-D signals, which is capable of considerably reducing the spectral redundancy while suppressing the least amount of spectral energy. Spectral characteristics of the signal, such as spectral extension shape and spectral occupancy, are taken into account for determining both decimation grid and antialiasing filter. In

particular, the spectral extension has been modelled as a second-order energy distribution through its principal axes and inertia moments of the power spectrum. This choice corresponds to approximating the spectral energy distribution with an ellipse, whose principal axes correspond to the radii of gyration of the spectrum, and allows us to considerably simplify the structure of the prefilter. As a consequence, the procedure for jointly designing subgrid and prefilter becomes purely geometrical and of immediate application.

We have implemented our decimation technique into a fully-automated computer procedure which, after having analysed the spectral content of a discrete 2-D signal, generates all sublattices that are compatible with its spectral extension, finds the corresponding fundamental cells, and selects the prefilter/subgrid pair that best fits the estimated spectral extension.

The proposed method has been shown to be quite effective for reducing the spectral redundancy of a 2-D signal. After having applied it on a series of test images with various spectral content, we have shown that we can typically reach decimation ratios that range from 10 to 20 with an acceptable loss of quality after reconstruction. It should be quite clear, however, that the 2-D signals that are appropriate the most for the proposed decimation technique are those whose spectral extension is well-described by a second-order model. Signals with more complex spectral content might give rise to a more modest results in the reconstruction quality. For example, the method might not perform at its best with signals having a noncompact spectrum. Nevertheless, the ideas presented in this article might be adapted to different spectral geometries by modifying the spectral extension model and the geometrical procedure for synthesising the relative prefilter.

To estimate the second-order spectral extension associated to a 2-D signal the technique proposed in this article needs to perform the computation of a two-dimensional FFT. The computational load associated to this operation is quite heavy; therefore, it would be worth investigating the possibility of estimating the spectral extension through the analysis of the signal

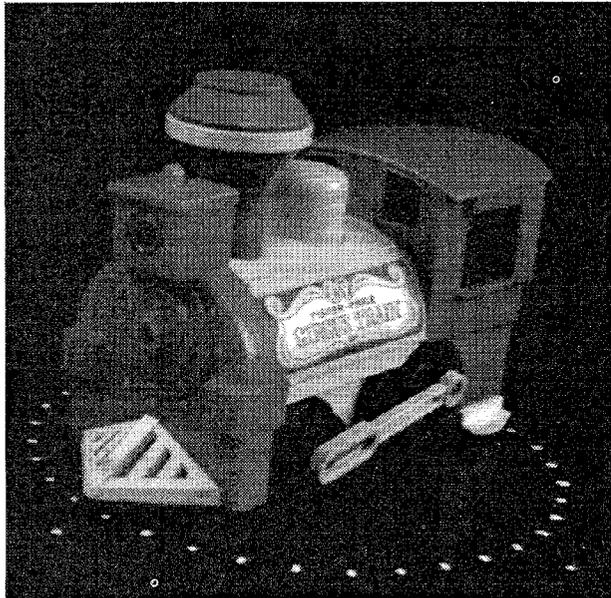


Fig. 13 Reconstructed test image 2



a



b

Fig. 14 Reconstruction results for test image 2 with the 12th order decimation basis $v_1 = [3 \ 0]^T$, $v_2 = [1 \ 4]^T$
 a Zoomed-in detail of original image
 b Zoomed-in detail of reconstructed image

rather than that of its Fourier transform. One solution that we are currently investigating uses the projection-slice theorem for estimating the spectrum extension model through a limited number of 1D FFTs. We are also working on the implementation of the hexagonal filters as the cascade of three properly rotated one-dimensional low-pass filters.

Note that the proposed method analyses the signal in whole, as if it were stationary. Only uniform decimation grids are, in fact, considered. The assumption of stationarity, however, is generally not correct for images, for which it would be better to assume region-wise stationarity. Indeed, it is reasonable to expect a region-wise adaptive implementation of the algorithm to outperform the proposed one. The derivation of a region-wise implementation, however, would involve a number of problems, such as those that filtering operations would cause in the proximity of region borders, or those connected with the estimation of the spectral extension parameters in irregular regions of small size.

6 References

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7 Appendix

7.1 Glossary

Z, Q, R = sets of integer, rational and real elements, respectively

Z_M, Q_M, R_M = $M \times M$ matrices with integer, rational and real elements, respectively

$A_{i,j}$ = (i, j) th element of the matrix A

A^{-T} = $(A^{-1})^T$ = transposed of the inverse of the matrix A

$Z_{M,n}$ = $\{A \in Z_M, |\det(A)| = n\}$

$\mathcal{H}_{M,n}$ = set of matrices of $Z_{M,n}$ in Hermite normal form

7.2 Hermite normal form

A nonsingular matrix $\mathbf{A} \in Z_M$ is said to be *unimodular* if $\mathbf{A}^{-1} \in Z_M$. The necessary and sufficient condition for \mathbf{A} to be unimodular is $|\det(\mathbf{A})| = 1$. Two matrices $\mathbf{A}, \mathbf{B} \in Z_M$ are said to be *right-equivalent* if there exists a unimodular matrix $\mathbf{V} \in Z_M$ such that $\mathbf{B} = \mathbf{AV}$. Left-equivalence is similarly defined. Two matrices $\mathbf{A}, \mathbf{B} \in Z_M$ are said to be *equivalent* if there exist two unimodular matrices $\mathbf{U}, \mathbf{V} \in Z_M$ such that $\mathbf{B} = \mathbf{UAV}$.

Two equivalent (or just right/left-equivalent) matrices always have the same *modulus* (absolute value of their determinant). As a consequence, given a positive integer n , it is always possible to uniquely subdivide the set $Z_{M,n} = \{\mathbf{A} \in Z_M, |\det(\mathbf{A})| = n\}$ into equivalence classes. The set of equivalence classes represents a partition of $Z_{M,n}$.

Definition 1: A matrix $\mathbf{A} \in Z_{M,n}$ is said to be in Hermite normal form [13] if

- (i) \mathbf{A} is upper triangular
- (ii) $A_{ij} \geq 0$
- (iii) $A_{ij} < A_{ii}$, $1 \leq i < j \leq M$, if $A_{ii} \neq 0$
- (iv) $A_{ij} = 0$ if $A_{ii} = 0$.

The number of distinct matrices of $Z_{M,n}$ in Hermite normal form is thus given by the sum of all possible distinct integer factorisations of n .

Theorem 1 (Hermite normal form theorem): Every nonsingular matrix of Z_M is the right-equivalent of one and only one matrix in Hermite normal form.

Such result [13] is of fundamental importance as it implies that each class of right-equivalents in $Z_{M,n}$ contains a unique matrix in Hermite normal form.