Toward Nonlinear Wave Digital Filters

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***Abstract*— The wave digital filter (WDF) theory provides us with a systematic methodology for building digital models of analog filters through the discretization of their individual circuit components. In some situations, WDF principles can also be successfully used for modeling circuits in which a nonlinear circuit element is present under mild conditions on its characteristic.**

In this paper, we propose an extension of the classic WDF principles, which allows us to considerably extend the class of nonlinear elements that can be modeled in the wave digital domain. The method we propose is based on a new class of waves that can be chosen in such a way that incorporates the intrinsic dynamics of a nonlinear element into a new class of dynamic multiport adaptors. This family of junctions represents a generalization of the concept of "mutator" in the analog nonlinear circuit theory because it allows us to treat a nonlinear dynamic element as if it were instantaneous (resistive).

I. Introduction

WAVE digital filters (WDF's) [1]–[3] are well known for possessing many desirable properties over other digital filter implementations [4]. In fact, not only are WDF structures designed after analog (classical) circuits, but they tend to preserve most of the good properties of their analog counterpart. For example, passivity and losslessness of analog filters are preserved by their wave digital implementation [5]. Furthermore, the behavior of WDF's is less sensitive to the quantization of the coefficients; therefore, WDF's exhibit modest accuracy requirements without giving up good dynamic range performance. The sensitivity properties of WDF's also guarantee stability under mild conditions, producing WDF structures that exhibit neither limit cycles nor zero-input parasitic oscillations [6].

The WDF theory for the synthesis of linear filters has reached, over the past two decades, an advanced level of maturity. In fact, a large variety of WDF-based techniques has been developed for a wide range of applications [1]. More recently, however, we witnessed renewed interest in WDF's as the research in musical acoustics started to turn toward synthesis through *physical modeling* [7]–[10].

Over the past few years, a variety of applications aimed at the physical modeling of musical instruments or acoustic environments has been developed. Some of these solutions are based on a model description that is based on scattering parameters and wave variables. In particular, some of the physical models that are available in the literature consist of an interconnection of the typical building blocks of WDF's and digital waveguides [10] (DWG's). These latter elements can be seen as the *distributed-parameter* counterpart of the (*lumped*) WDF blocks. DWG elements are particularly suitable for modeling acoustic resonating structures that are fully compatible with WDF's.

Hybrid WDF/DWG structures represent a good solution to the problem of sound synthesis by physical modeling because, besides referring to an acoustic instrument, they are based on a *local* (block-based) discretization of the physical elements that constitute the analog model. In other words, there seems to be potential for a flexible synthesis approach based on the interconnection of predefined building blocks.

One major problem of sound synthesis techniques based on physical modeling, however, is still the treatment of nonlinearities, which are predominant in musical acoustics. They, in fact, are the main responsible of the timbral dynamics of the instruments; therefore, they cannot be modeled through simple linearization. In fact, while the WDF theory is a well-established theory, the wave digital (WD) theory of nonlinear circuits is still far from being formalized and developed in a homogeneous way. As a matter of fact, the tools and methods developed for the linear theory are often not sufficient for covering most of the nonlinear cases, particularly when the nonlinearity is described by a differential equation rather than an algebraic equation, as in the resistive case.

Among the results on nonlinear WD structures that are available in the literature, it is important to mention Meerkötter's work [11] on WD circuits that contain a nonlinear resistor. In that work, the transformation that defines pairs (a, b) of waves as a function of Kirchhoff pairs (v, i) of variables (voltage and current) is used for mapping the characteristic of a nonlinear resistor onto the WD domain, i.e., for transforming a $v - i$ curve into a $b - a$ curve. Such a WD characteristic can be inserted into a WDF structure by connecting it to a reflection-free (adapted) port of the circuit. This WD mapping approach, however, requires the nonlinear element to be resistive. In this situation, in fact, the element is described by an algebraic relationship between voltage and current that is to be rewritten as an explicit relationship in the WD domain (wave b as a function of wave a).

The nonlinear resistors represent only a subset of the so-called "algebraic" nonlinearities [12]. Algebraic bipoles are described by an equation between the two port variables $v^{(j)}$ and $i^{(k)}$, where $j, k \in \{0, \pm 1, \pm 2, \cdots\}$ denote time-differentiation (if positive) or integration (if negative) of v and i. Nonlinear devices that are not algebraic are called *dynamic* [12] elements. Examples of nonlinear algebraic bipoles are

Manuscript received October 24, 1996; revised September 1, 1998. The associate editor coordinating the review of this paper and approving it for publication was Prof. Peter C. Doerschuk.

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Publisher Item Identifier S 1053-587X(99)03655-7.

nonlinear resistors, capacitors, inductors, FNDR's, supercapacitors, superinductors, memristors, etc.

Modeling nonresistive algebraic nonlinearities with classical WDF principles is known to give rise to problems of computability because closed loops without delays cannot be avoided in the resulting WD structure. An example of WD implementation of a circuit containing a nonlinear reactance can be found in [13]. The proposed approach consisted of linearizing the bipole through an ideal transformer whose turns ratio depends on the local terminal voltage. For example, a nonlinear capacitance is modeled as a linear capacitance connected to the output of a variable transformer. This solution can be easily proven to preserve the losslessness of the nonlinear capacitance. However, as we can expect, the fact that the turns ratio depends on the port voltage gives rise to a problem of computability of the resulting scheme. In fact, the voltage that the turns ratio depends on can only be derived numerically by solving a nonlinear implicit equation at every time instance.

In similar situations, other authors [14] chose a more rudimentary solution that consists of inserting a delay element where the noncomputable connection (delay-free loop) is found. This solution, however, could easily introduce unacceptable discretization errors or instability problems.

In order to overcome computability problems without having to solve implicit equations, a different solution for a wave implementation of circuits that contain reactive nonlinearities was proposed in [15] and [16]. In this solution, new waves are defined in order to be suitable for the direct modeling of algebraic nonlinearities such as capacitors and inductors. In fact, with respect to the new waves, the description of the nonlinear element becomes purely algebraic so that the results already formulated for nonlinear resistors [11] can be applied. In order to adopt such new waves, a special two-port element that performs the change of variables is defined and implemented in a computable fashion. The reactive nonlinear element is thus modeled in a new WD domain, where its description becomes memoryless. Roughly speaking, with respect to the new wave variables, the behavior of the nonlinear bipole becomes resistor-like, and therefore, the two-port junction that performs the change of wave variables plays the role of a device that transform the reactance into a resistor.

The above idea of transforming reactances into nonlinear resistors is not new in the theory of circuit design. In fact, the literature on nonlinear circuits is rich with results that allow the designer to model arbitrary nonlinear networks by using just nonlinear resistors, operational amplifiers, and other linear circuit elements [17]–[20]. By doing so, it is possible to design arbitrary bipoles without ever using a nonlinear inductor or a nonlinear capacitor, which are more difficult to implement. This is possible by using special two-port analog devices called *mutators* [12], [18], [17], which are built using only operational amplifiers and linear passive resistors and capacitors. In general, mutators reduce the problem of realizing a wide class of nonlinear bipoles with memory to that of synthesizing a nonlinear resistor. The method proposed in [15] and [16] is the digital counterpart of this analog approach to the design of nonlinear circuits.

In this paper, we further extend the ideas introduced in [15] and [16]. In particular, we propose a more general family of digital waves that allow us to model a wider class of algebraic and dynamic nonlinearities. The consequent generalization of the WDF principles include dynamic multiport junctions and adaptors, which synergetically combine ideas of nonlinear circuit theory (mutators) and WDF theory (adaptors). We will show that this generalization provides us with a certain degree of freedom in the design of WD structures. In fact, not only can we design a dynamic adaptor in such a way as to incorporate the whole dynamics of a nonlinear element into it, but we can also design a dynamic adaptor that will incorporate an arbitrarily large portion of a linear circuit.

We will show that under mild conditions on their parameters, such dynamic multiport adaptors are nonenergic, and therefore, the global stability of the reference circuit is preserved by the wave digital implementation. For this reason, such multiport junctions can be referred to as dynamic *adaptors*.

In order for this paper to remain as self-contained as possible, Section II is devoted to some of the basic concepts of the WDF theory (Section II-B) and the treatment of instantaneous nonlinearities in the WD domain (Section II-C). The generalization of WDF theory is presented in Section III. In particular, in Section III-A, a new general definition of digital waves is introduced, and its impact on the structure of scattering junctions is presented in Section III-B. A specific class of elementary scattering junctions with memory that represent the WD equivalent of mutators is then presented in Section III-C, whereas the dynamic multiport junctions are discussed in Section III-D. The problem of the passivity of the dynamic multiport junctions is discussed in Section III-E. Finally, some examples of applications are developed and discussed in Section IV, where simulation results are presented as well.

Throughout the paper, we will indicate with lowercase letters the signals in the time domain, whereas the corresponding capital letters will denote either their Laplace transform (when they are a function of the complex variable s) or their Z transform (if they depend on the complex variable z). Whenever the context does not help in distinguishing between them, it will be explicitly specified.

II. Preliminaries

In this section, the basics of WDF theory and some results on the treatment of nonlinear resistors are presented. Readers who are already familiar with concepts and tools of WDF theory may skip this part, whereas those who want to know more about it may refer to the work by Fettweis [1], [2], [4], [5] and Meerkötter [3], [6], [11]. For a more complete bibliography on the matter, see [1].

A. The Issue of Local Discretization

Constructing a digital model of a given analog filter can be quite easily performed by applying a bilinear mapping to the complex variable of its transfer function. This approach to the synthesis can be thought of as "global" as it considers the transfer function as a whole. Conversely, a "local" approach to discretization consists of properly interconnecting individually discretized portions of an analog filter. Although discretizing smaller portions of an analog filter is generally simpler, this way of proceeding may easily give rise to problems at the interconnection of the building blocks.

Let us consider, for example, the interconnection of two portions ($\mathcal{C}_1$ and $\mathcal{C}_2$) of an analog linear circuit. Each one of the two subcircuits can be seen as a set of linear differential equations (local description). The two subcircuits interact with each other through a port connection, which forces the port variables v and i to satisfy some continuity constraints (global constraints forced by the Kirchhoff equations). As a consequence, connecting the two subcircuits through a port means adding a set of "global" constraints to two "local" descriptions. If we individually discretized $\mathcal{C}_1$ and $\mathcal{C}_2$, we would obtain two digital filters $\mathcal{D}_1$ and $\mathcal{D}_2$ that, in general, cannot be directly connected together, as this would mean ignoring the global constraints. Furthermore, the output of each one of them would generally depend instantaneously on its input, and therefore, the interconnection of $\mathcal{D}_1$ and $\mathcal{D}_2$ could easily give rise to a delay-free loop (an implicit equation), which cannot be implemented "as is." Graphs that contain delay-free loops are said to be *noncomputable*.

Computability problems can be avoided by eliminating the instantaneous input–output connection in just one of the ports involved in each delay-free loop. This, unfortunately, is not possible when using Kirchhoff variables (voltage and current) as they are all instantaneously dependent on each other. A solution to this problem, which has been proposed by Fettweis in the late 1960's [2], consists of adopting a new set of variables that can be obtained from the Kirchhoff variables through a linear invertible transformation. The definition of such variables resembles that of traveling waves in electrical lines. Therefore, the concept of *adaptation* can be successfully used to avoid instantaneous input/output dependencies (reflections) where needed and make the implementation scheme computable. Such filters, known as WDF's, represent a discrete implementation of analog linear filters that are obtained through the interconnection of individually discretized circuit elements and all described by wave variables.

B. Some Notes on Classical WDF Theory

Let us consider two Kirchhoff variables v and i that characterize any port of a linear circuit. Such variables can be mapped onto a new pair of variables a and b by means of a linear transformation of the form

$$a = v + Ri, \quad b = v - Ri \tag{1}$$

parametrized by a *reference resistance* R. The variables a and b can be thought of as *waves* that travel in opposite directions through an infinitesimal transmission line of characteristic impedance R. If $R \neq 0$, then the above linear transformation is invertible; therefore, there is no loss of information in using wave variables for describing a Kirchhoff pair (v, i). As a circuit port is electrically characterized by a Kirchhoff pair, its *wave* representation will be given in terms of the waves a (input) and b (output).

A linear resistor R_1 can be described in the WD domain (a, b) by a relationship of the form $b = ka$, where $k = (R_1 - R)/(R_1 + R)$. By choosing $R = R_1$ (adaptation), the *reflection coefficient* k becomes zero, and so does the "reflected" (output) wave, no matter what the "incident" (input) wave is. The parameter R can thus be chosen in such a way to eliminate the instantaneous input/output dependency.

When the bipole is an ideal capacitor C, the relationship between Kirchhoff variables is a differential equation of the form $i = C\dot{v}$, which becomes algebraic in the domain of the Laplace transform $I(s) = sCV(s)$. By using (1) and the bilinear transformation [1], this relationship takes the form $B(z) = K(z)A(z)$, where

$$K(z) = \frac{p + z^{-1}}{1 + pz^{-1}}, \quad p = \frac{T - 2\tau}{T + 2\tau}$$

is an all-pass *reflection filter*, where T is the sampling interval, and $\tau = RC$ is a time constant. Once again, the parameter R can be chosen in such a way to eliminate the instantaneous input/output dependency. In fact, by letting $R = T/(2C)$, we have $K(z) = z^{-1}$.

The case of the linear inductor is similar to that of the linear capacitor. In fact, its WD description is an allpass filter whose transfer function, except for a sign change, is the same as that of the capacitors, with $\tau = L/R$. The condition is thus given by $R = 2L/T$, in which case, we have $K(z) = -z^{-1}$.

In order to obtain a WD implementation of a circuit, not only do we need a wave description of the individual elements, but we need to specify the topology of their interconnection. Circuit topology is given in terms of the Kirchhoff equations at all circuit nodes and loops and is implemented by means of multiport junctions called *adaptors*.

The two fundamental types of interconnection (parallel and series) are sufficient for specifying the circuit topology. Their wave equivalents, i.e., parallel and series adaptors, by implementing the corresponding Kirchhoff laws, act as an interface between wave pairs that are referred to different reference resistances.

For example, a parallel connection of n ports with port conductances $G_1 = 1/R_1$ to $G_n = 1/R_n$ is characterized by the Kirchhoff equations $v_1 = v_2 = \cdots = v_n$ and $i_1 + \cdots + i_n = 0$. The mth output wave $b_m, m = 1, \cdots, n$ can be written as a function of all input waves $a_k, k = 1, \cdots, n$ as $b_m = a_0 - a_m$, where $a_0 = \gamma_1 a_1 + \cdots + \gamma_n a_n$ and $\gamma_k = 2G_k/G, G = G_1 + \cdots + G_n$ are the reflection coefficients. Notice that $\gamma_1 + \cdots + \gamma_n = 2$; therefore, the degrees of freedom of an n-port parallel junction are down to $n-1$, which makes one of the n ports dependant on the others. In particular, we may choose the reference conductances in such a way that, for example, $\gamma_n = 1$. This happens when $G_1 + \cdots + G_{n-1} = G_n$, in which case, b_n becomes independent of a_n, and port n becomes reflection free.

A multiport junction whose port resistances are chosen so that the reflection coefficient of one port is zero is called a *reflection-free* adaptor, and the port that exhibits no input/output connection is called an *adapted* port. In all other cases, the adaptor is said to be *unconstrained*. The series adaptor can be derived in a similar way.

By exploiting the adaptation conditions at junctions and bipoles, it is possible to build a computable WD structure for modeling any linear circuit.

It is important to emphasize the fact that multiport adaptors are nonenergic [1] as the steady-state pseudo-power absorbed by them is always zero.

For a detailed summary of all standard WDF multiport junctions, see [3] and [4].

C. Nonlinear Resistors

A nonlinear resistor is generally defined as an algebraic relationship of the form $F(v,i)=0$ in the Kirchhoff variables v and i. The *Kirchhoff* characteristic of the resistor can be transformed into a *wave* characteristic of the form $f(a,b)=0$ through the following mapping:

$$f(a,b)=F\left(\frac{a+b}{2},\frac{a-b}{2R}\right).$$

The conditions that allow us to write the reflected wave b as an explicit function $b=\tilde{f}(a)$ of wave a are provided by the *implicit function theorem*. In fact, if the characteristic function $f(a,b)$ of the resistor is continuous with its derivatives and if the point (a_0,b_0) lies on the characteristic of the resistor [i.e., if $f(a_0,b_0)=0$], then the condition

$$\left.\frac{\partial f}{\partial b}\right|_{(a_0,b_0)}\neq 0$$

guarantees the existence of a function $\tilde{f}(\cdot)$ such that $f(a,\tilde{f}(a))=0$ in a neighborhood of a_0.

For example, the nonlinear characteristic $F(v,i)=v-v(i)=0$ of a current-controlled nonlinear resistor is mapped onto the wave characteristic

$$f(a,b)=\frac{a+b}{2}-v\left(\frac{a-b}{2R}\right)=0.$$

Therefore, we have

$$\begin{aligned}\frac{\partial f}{\partial b}&=\frac{\partial}{\partial b}\left\{\frac{a+b}{2}-v\left(\frac{a-b}{2R}\right)\right\}\\&=\frac{1}{2}-\frac{\partial v}{\partial i}\,\frac{\partial i}{\partial b}=\frac{1}{2}+\frac{1}{2R}\,v'\left(\frac{a-b}{2R}\right)\end{aligned}$$

where $v'(i)=(dv/di)$.

In conclusion, the *local invertibility* of the characteristic $v=v(i)$, i.e., the possibility of rewriting it in the form $b=\tilde{f}(a)$, is guaranteed by the condition $v'(i)\neq -R$.

A similar procedure can be followed for a voltage-control resistor with characteristic $F(v,i)=i-i(v)$. In this case, the local invertibility condition is given by $i'(v)\neq -1/R$.

The above results can be extended to the case in which the nonlinear characteristic of the resistor is only continuous piecewise linear [11]. For example, the invertibility of the characteristic $i=i(v)$ of a voltage-controlled resistor is guaranteed by

$$\inf_{v_2\neq v_1}\frac{i(v_2)-i(v_1)}{v_2-v_1}>-\frac{1}{R}$$

or

$$\sup_{v_2\neq v_1}\frac{i(v_2)-i(v_1)}{v_2-v_1}<-\frac{1}{R}.$$

If such invertibility conditions are satisfied, a nonlinear resistance can be implemented in the wave domain as a nonlinear map of the form

$$b=\tilde{f}(a) \tag{2}$$

to be connected to an *adapted* port of the WDF structure in order to avoid computability problems.

III. GENERALIZATION

As previously stated in the introduction, the nonlinear elements that are most often encountered in nonlinear circuit theory are those that belong to the class of "algebraic" nonlinearities [12]. In the one-port case, such elements are characterized by an algebraic relationship between two port variables $v^{(j)}$ and $i^{(k)}$, where $j,k\in\{0,\pm1,\pm2,\cdots\}$ denote time-differentiation (if positive) or integration (if negative) of v and i. A multiport algebraic element is defined as an algebraic relationship between two such variables for each port. All nonlinear devices that are not algebraic are called *dynamic* [12] elements.

Modeling an algebraic bipole directly in terms of the waves (1) can only be done when it is defined as an algebraic relationship between voltage and current at the port of connection (resistive bipole). When the nonlinearity involves derivatives and/or integrals of such variables (e.g., nonlinear inductors and capacitors), any modeling attempt based on classical WD principles fails because of computability problems. In this case, the implicit equation corresponding to the closed loop needs to be solved at every time instance.

In order to overcome this difficulty, new types of waves have been proposed in the literature [15], [16]. Through such new waves, a nonlinear reactance can be treated as if it were resistive. This solution is inspired by a method that is widely used in nonlinear circuit theory for implementing a wide class of nonlinear bipoles with memory, including nonlinear capacitors and inductors. The method is based on special two-port analog elements [12], [17], [18] called *mutators*, which are built with only operational amplifiers and linear passive resistors and capacitors. Such devices are used, for example, to "mutate" a nonlinear resistor into a nonlinear inductor while preserving its nonlinear characteristic in the transformed Kirchhoff domain.

In this section, we will show that a wider class of digital waves can be defined, and a general family of WD mutators can be introduced and adopted for modeling nonlinear algebraic devices and some dynamic nonlinearities.

A. Wave Variables with Memory

Let us consider the analog Kirchhoff variables $v(t)$ and $i(t)$ that characterize a circuit port. The Laplace transformation, followed by bilinear transformation, will provide us with the Z-transform of the discretized versions of such signals. Now, instead of defining a pair of waves through the usual definition

(1), which refers to an arbitrary port resistance R, we define a new pair of wave variables in the Z-domain, with reference to an arbitrary port "impedance" (transfer function) $R(s)$, as

$$A(z) = V(z) + R(z)I(z), \quad B(z) = V(z) - R(z)I(z). \quad (3)$$

The port impedance $R(z)$ will be referred to as a *reference transfer function* (RTF). In order to explain the consequences of the above definition, we will now derive the description of generic linear and nonlinear bipoles in the extended WD domain.

1) Linear Bipoles: In order to clarify how the above definition of waves can be used for extending the validity of WD principles to a wider class of nonlinear elements, let us first consider the case of a linear time-invariant macro-bipole, characterized by a relationship of the form $V(s) = R_1(s)I(s)$, in the domain of the Laplace transform. A linear bipole of this type can, in fact, be a whole linear circuit as seen from any of its ports. By adopting the new pair of waves (3) and using the bilinear transformation, the WD version of this bipole assumes the form $B(z) = K(z)A(z)$, where

$$K(z) = \frac{R_1(z) - R(z)}{R_1(z) + R(z)}$$

takes on the meaning of a reflection filter with transfer function $K(z)$, which must be guaranteed to be causal and stable through a proper choice of the RTF $R(z)$.

When the RTF matches the transfer function of the linear bipole ($R(z) = R_1(z)$), the reflected wave becomes zero ($K(z) = 0$), and the dynamics of the linear bipole is now embodied into the wave pair (a, b). This condition of *total adaptation* $R(z) = R_1(z)$, however, is a very strong one. Since the main problem to avoid is that of the delay-free loops in the interconnection of wave elements, what will actually be required in most cases is the elimination of just the *instantaneous* portion of the reflected wave (*instantaneous adaptation*). In order to prevent the wave representation of the linear bipole from instantaneously "reflecting" the "incident" wave, we need the reflection filter $K(z)$ to exhibit no instantaneous input/output connection, i.e., $K(z) = z^{-1}\hat{K}(z)$, with $\hat{K}(z)$ causal and stable. When the RTF and the bipole transfer function are rational functions of z of the form

$$R(z) = \frac{C(z)}{D(z)} = \frac{c(0) + \sum_{i=1}^{N} c(i)z^{-i}}{1 + \sum_{i=1}^{M} d(i)z^{-i}}$$

$$R_1(z) = \frac{C_1(z)}{D_1(z)} = \frac{c_1(0) + \sum_{i=1}^{N'} c_1(i)z^{-i}}{1 + \sum_{i=1}^{M'} d_1(i)z^{-i}} \quad (4)$$

then the absence of an instantaneous reflection at the bipole port is guaranteed by the condition $c(0) = c_1(0)$ (*instantaneous adaptation*). However, many *intermediate* conditions of adaptation are possible (*partial adaptation*), depending on how simple we would like the result of the reflection filter $K(z)$ to be. In conclusion, the RTF can be chosen in quite an arbitrary fashion, provided that the stability condition on K is satisfied. The only constraint that could be required for avoiding delay-free loops is on the leading terms $c(0)$ and $c_1(0)$ of the transfer functions.

2) Filtered Algebraic Nonlinearities: Let us now consider a nonlinear bipole characterized by the equation $F(v, q) = 0$, where $q(t)$ is a filtered version of $i(t)$. More precisely, q and i are bound to satisfy a differential equation that, after Laplace transformation, assumes the form $Q(s) = H_i(s)I(s)$. Notice that when $H_i(s)$ is a constant, the nonlinear element becomes a nonlinear resistor (see Section II-C). More generally, when $H_i(s) = \beta s^k$, with k integer and β constant, then the bipole becomes algebraic; otherwise, the bipole is dynamic.

As H_i is a function of s, the nonlinear element cannot be considered as instantaneous with respect to v and i, but it can still be considered as memoryless with respect to the Kirchhoff pair (v, q), which means that we can use the results of Section II-C on nonlinear resistors [11], provided that we define a wave pair of the form $a = v + \mu q, b = v - \mu q, \mu$ being a constant reference parameter. With this choice, the wave characteristic of the nonlinear element becomes

$$F(v, q) = F\left(\frac{a+b}{2}, \frac{a-b}{2\mu}\right) = f(a, b) = 0$$

which, under proper conditions of the type specified in Section II-C, can be expressed in explicit form.

Let $H_i(z)$ be the result of the discretization of $H_i(s)$ obtained through bilinear transformation. The new pair of digital waves can thus be defined as

$$A(z) = V(z) + \mu Q(z) = V(z) + \mu H_i(z)I(z)$$
$$B(z) = V(z) - \mu Q(z) = V(z) - \mu H_i(z)I(z)$$

where $\mu \neq 0$. Such waves correspond to (3), with $R(z) = \mu H_i(z)$, and they are chosen in such a way as to incorporate the dynamics of the nonlinear bipole so that the nonlinearity can be treated as memoryless.

Notice that, although voltage and charge can be directly computed from such waves as

$$v = \frac{a+b}{2}, \quad q = \frac{a-b}{2\mu}$$

the current can only be derived through filtering as

$$I(z) = \frac{A(z) - B(z)}{2\mu H_i(z)}.$$

This means that depending on which RTF is being considered, the above waves can be attributed different interpretations.

A more general class of dynamic nonlinearities is represented by bipoles whose characteristic is of the form $F(p, q) = 0$, where p and q are filtered versions of v and i, respectively. Let $P(s) = H_v(s)V(s)$ and $Q(s) = H_i(s)I(s)$, where $H_v$is the *voltage filter* and H_i the *current filter*. Nonlinear resistors are obtained with filters with constant frequency response. All algebraic bipoles are modeled by letting $H_v(s) = \alpha s^j$ $H_i(s) = \beta s^k$ with α and β constants and j and k integers. In all other cases, such nonlinearities are dynamic. As we can see, this class of nonlinearities is wider than the previous one as, besides including all algebraic bipoles, it covers all those nonlinear devices that can be represented as a cascade of two filters with an instantanous nonlinear element in between.

Like in the previous case, the nonlinearity results as being memoryless with respect to pair of waves $a = p + \mu q, b = p - \mu q, \mu$ being, once again, a reference parameter. Let $H_v(z)$ be the result of the discretization of $H_v(s)$ obtained through bilinear transformation. The new pair of digital waves can be defined as

$$\begin{aligned} A(z) &= P(z) + \mu Q(z) = H_v(z)V(z) + \mu H_i(z)I(z) \\ B(z) &= P(z) - \mu Q(z) = H_v(z)V(z) - \mu H_i(z)I(z). \end{aligned} \quad (5)$$

Notice that this time, both Kirchhoff variables v and i must be derived from the waves a and b through the filtering of

$$V(z) = \frac{A(z) + B(z)}{2H_v(z)}, \qquad I(z) = \frac{A(z) - B(z)}{2\mu H_i(z)}.$$

However, p and q can be directly computed from the above waves as $p = (a + b)/2$ and $q = (a - b)/(2\mu)$. As a consequence, under proper invertibility conditions, the wave characteristic of the nonlinear bipole

$$F(p, q) = F\left(\frac{a + b}{2}, \frac{a - b}{2\mu}\right) = f(a, b) = 0$$

can be expressed in the form $b = \tilde{f}(a)$.

B. Scattering

We now consider the problem of how to perform a transformation between two different pairs of digital waves of the types defined in the previous section. This operation, in fact, can be helpful for understanding the structure of multiport junctions, which are the basic elements for building wave digital structures. We will first examine the digital waves of (3), which will be referred to as *single-filter waves*. These results will then be extended to the more general waves of (5), which we will refer to as *double-filter waves*.

Single-Filter Waves: A scattering junction is a two-port element whose aim is that of transforming the wave pair (a_1, b_1), which is referred to the RTF $R_1(z)$, into the wave pair (a_2, b_2), which is referred to $R_2(z)$. Let

$$\begin{aligned} A_1(z) &= V_1(z) + R_1(z)I_1(z) \\ B_1(z) &= V_1(z) - R_1(z)I_1(z) \\ A_2(z) &= V_2(z) + R_2(z)I_2(z) \\ B_2(z) &= V_2(z) - R_2(z)I_2(z) \end{aligned}$$

where $A_1(z)$ and $A_2(z)$ are the waves that enter the junction, and $B_1(z)$ and $B_2(z)$ the corresponding reflected waves. The scattering junction is then characterized by the continuity constraints $V_1(z) = V_2(z)$ and $I_1(z) + I_2(z) = 0$.

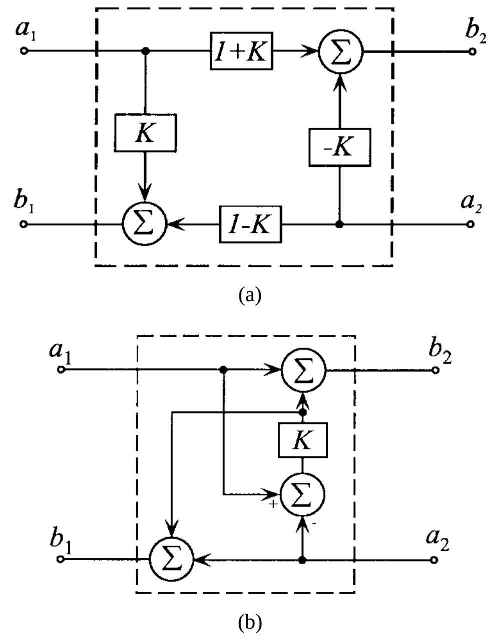

Fig. 1. Direct implementation (a) and one-filter implementation (b) of the scattering junction.

The output waves can be easily expressed as a function of the input waves as

$$\begin{aligned} B_1(z) &= K(z)A_1(z) + (1 - K(z))A_2(z) \\ B_2(z) &= (1 + K(z))A_1(z) - K(z)A_2(z) \end{aligned} \quad (6)$$

where

$$K(z) = \frac{R_2(z) - R_1(z)}{R_2(z) + R_1(z)}$$

is the transfer function of the *reflection filter* that characterizes the scattering junction with memory, which is expected to be causal and stable. Properties of passivity and losslessness will be discussed at the end of this section.

A direct implementation of (6) is shown in Fig. 1(a), which can also be implemented with just one scattering filter, as shown in Fig. 1(b). From Fig. 1, we notice that the wave a_1 that enters port 1 is partially reflected through the filter $K(z)$ and partially transmitted to port 2 through the filter $1 + K(z)$, whereas the wave a_2 entering port 2 is partially reflected through the filter $-K(z)$ and partially transmitted through the filter $1 - K(z)$.

Notice that the above two-port junction is suitable for modeling the scattering between all types of *dimensionally homogeneous* one-filter waves, i.e., not just waves of voltage dimension but any type of wave pairs of the same dimension.

When considering the digital implementation of a scattering junction, it is of crucial importance to derive the conditions under which any of its ports do not exhibit any instantaneous reflection, as its interconnection with other circuit ports might give rise to delay-free loops. In order to avoid instantaneous reflection of the waves entering the two ports of a digital scattering junction, it is necessary and sufficient for $K(z)$ to exhibit no instantaneous input/output connection, i.e., $K(z) = z^{-1}\hat{K}(z)$, with $\hat{K}(z)$ causal and stable.

When the two reference impedances are rational functions of z

$$R_1(z) = \frac{C_1(z)}{D_1(z)} = \frac{c_1(0) + \sum_{i=1}^{N_1} c_1(i) z^{-i}}{1 + \sum_{i=1}^{M_1} d_1(i) z^{-i}}$$

$$R_2(z) = \frac{C_2(z)}{D_2(z)} = \frac{c_2(0) + \sum_{i=1}^{N_2} c_2(i) z^{-i}}{1 + \sum_{i=1}^{M_2} d_2(i) z^{-i}} \tag{7}$$

the condition of instantaneous adaptation is the same as that of linear bipoles (see Section III-A.1), i.e., $c_1(0) = c_2(0)$. Total adaptation $R_1 = R_2$ is also possible and represents the case in which the scattering junction becomes a direct connection between the two ports. Finally, partial adaptation is a condition that lies anywhere in between instantaneous and total adaptation, depending on how simple we would like the reflection filter K to result. In conclusion, the RTF's can be chosen in quite an arbitrary way, provided that K is causal and stable. The only constraint that could be required for avoiding delay-free loops is on the leading terms $c_1(0)$ and $c_2(0)$ of the two RTF's.

Double-Filter Waves: We now want to model the scattering between waves of the form (5), i.e., waves that are dimensionally nonhomogeneous. In order to do so, let us consider the two pairs of digital waves (a_1, b_1) and (a_2, b_2) given by

$$\begin{aligned} A_1(z) &= H_{v_1}(z)V_1(z) + H_{i_1}(z)I_1(z) \\ B_1(z) &= H_{v_1}(z)V_1(z) - H_{i_1}(z)I_1(z) \\ A_2(z) &= H_{v_2}(z)V_2(z) + H_{i_2}(z)I_2(z) \\ B_2(z) &= H_{v_2}(z)V_2(z) - H_{i_2}(z)I_2(z) \end{aligned} \tag{8}$$

where, without loss of generality, we dropped the free parameter μ and apply the continuity constraints $V_1(z) = V_2(z) = V(z)$ and $I_1(z) = -I_2(z) = I(z)$ in order to express the "reflected" waves $B_1(z)$ and $B_2(z)$ as a function of the "incident" waves $A_1(z)$ and $A_2(z)$. In fact, by defining a reflection filter of the form

$$K(z) = \frac{H_{v_1}(z)H_{i_2}(z) - H_{v_2}(z)H_{i_1}(z)}{H_{v_1}(z)H_{i_2}(z) + H_{v_2}(z)H_{i_1}(z)} \tag{9}$$

we find two alternative sets of scattering equations, which lead to two different implementations. The first one is given by

$$\begin{aligned} B_1(z) &= K(z)A_1(z) + \frac{H_{v_1}(z)}{H_{v_2}(z)}(1 - K(z))A_2(z) \\ B_2(z) &= \frac{H_{v_2}(z)}{H_{v_1}(z)}(1 + K(z))A_1(z) - K(z)A_2(z) \end{aligned} \tag{10}$$

and can be implemented as shown in Fig. 2(a), whereas the second one is given by

$$\begin{aligned} B_1(z) &= K(z)A_1(z) + \frac{H_{i_1}(z)}{H_{i_2}(z)}(1 + K(z))A_2(z) \\ B_2(z) &= \frac{H_{i_2}(z)}{H_{i_1}(z)}(1 - K(z))A_1(z) - K(z)A_2(z) \end{aligned} \tag{11}$$

and can be implemented as shown in Fig. 2(b).

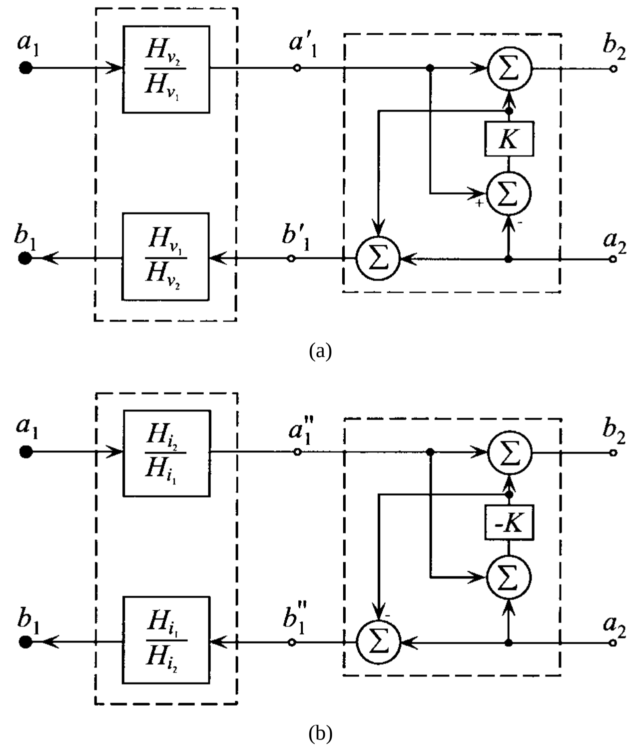

Fig. 2. Scattering two-port cell. Two equivalent implementations.

Notice that one of the two above structures might be preferable to the other one, depending on which one of the two filters $H_{v_1}(z)/H_{v_2}(z)$ and $H_{i_1}(z)/H_{i_2}(z)$ is the simplest. In fact, it often happens that either $H_{v_1}(z)/H_{v_2}(z)$ or $H_{i_1}(z)/H_{i_2}(z)$ results as being memoryless, in which case, a one-filter scattering junction can be employed. In addition, notice that it is possible to reverse the order in which the two two-port elements in dashed boxes of Figs. 2(a) and (b) are cascaded.

Notice that with reference to Fig. 2(a), (a′) and (b′) can be interpreted as one-filter waves of the form

$$\begin{aligned} A_1'(z) &= P_1(z) + R_1(z)Q_1(z) \\ B_1'(z) &= P_1(z) - R_1(z)Q_1(z) \end{aligned}$$

where $P_1(z) = H_{v_2}(z)V_1(z), Q_1(z) = H_{v_2}(z)I_1(z)$, and $R_1(z) = H_{i_1}(z)/H_{v_1}(z)$, which need to be interfaced with other compatible one-filter waves of the form

$$\begin{aligned} A_2(z) &= P_2(z) + R_2(z)Q_2(z) \\ B_2(z) &= P_2(z) - R_2(z)Q_2(z) \end{aligned}$$

where $P_2(z) = H_{v_2}(z)V_2(z)$, $Q_2(z) = H_{v_2}(z)I_2(z)$, and $R_2(z) = H_{i_2}(z)/H_{v_2}(z)$. In conclusion, the left dashed box of Fig. 2(a) acts as a *frequency-selective transformer* whose aim is to make the waves dimensionally homogeneous.

In addition, in this case, we can derive the conditions under which a port does not exhibit instantaneous reflection. Again, there is a certain freedom in the adaptation conditions. For example, we may require the reflection filter $K(z)$ of (9) to become identically zero by letting $H_{v_1}(z)H_{i_2}(z) = H_{v_2}(z)H_{i_1}(z)$. This condition of *total adaptation* is a rather restrictive one, but it can be relaxed by requiring $K(z)$ just to exhibit no instantaneous input/output connection, i.e., to be of the form $K(z) = z^{-1}\hat{K}(z)$, with $\hat{K}(z)$ causal and stable.

Let us assume the four causal filters $H_{v_1}(z), H_{v_2}(z), H_{i_1}(z)$, and $H_{i_2}(z)$ to be described by rational functions of the form of (7). A necessary and sufficient condition for avoiding instantaneous reflections at both ports of the scattering junction is

$$h_{v_1}(0)h_{i_2}(0) = h_{v_2}(0)h_{i_1}(0) \tag{12}$$

where $h_{v_1}(0), h_{v_2}(0), h_{i_1}(0)$, and $h_{i_2}(0)$ are the leading terms of the numerators of $H_{v_1}(z), H_{v_2}(z), H_{i_1}(z)$ and $H_{i_2}(z)$, respectively. In addition to the above adaptation condition, we need to make sure that $K(z)$ corresponds to a causal and stable filter.

C. Elementary Scattering Junctions: Mutators

A particular case of wave scattering junctions with memory is represented by the wave digital mutators [15], which are intimately related to a class of two-port analog elements [12], [17], [18] called *mutators*, which are built using only operational amplifiers and linear passive resistors and capacitors. Mutators can be used, for example, to transform a nonlinear inductor into a nonlinear resistor while preserving the nonlinear characteristic in the transformed Kirchhoff domain. This property can be quite useful as, for example, synthesizing a nonlinear inductor with a prescribed $\phi - i$ characteristic is generally much more difficult than implementing a nonlinear resistor with the same characteristic in the $v - i$ plane. In this case, an $R - L$ mutator can be used to map the $\phi - i$ plane onto the $v - i$ plane. In general, mutators reduce the problem of realizing nonlinear algebraic bipoles to that of synthesizing a nonlinear resistor.

The scattering junction with memory of the type defined in this section represents a direct extension of the concept of mutator because it is suitable for modeling a wide class of dynamic nonlinear elements (filtered algebraic nonlinearities), as explained in Section III-A2.

In this section, we consider the wave equivalents of the mutators that are used for modeling the simplest types of nonlinear elements with memory.

1) Nonlinear Capacitors—$R - C$ Mutator: The wave $R - C$ mutator is simply a scattering junction between a capacitive RTF and resistive one. With reference to the results of Section III-B, this situation can be dealt with by letting $R_1(s) = R$ and $R_2(s) = 1/(sC), C > 0$. After bilinear transformation, the scattering filter

$$K(z) = \frac{R_2(z) - R}{R_2(z) + R}, \quad R_2(z) = \frac{T}{2C}\,\frac{1 + z^{-1}}{1 - z^{-1}}$$

of Fig. 1 becomes a first-order causal allpass filter

$$K(z) = \frac{p + z^{-1}}{1 + pz^{-1}}, \quad p = \frac{T - 2RC}{T + 2RC}$$

which is stable when $|p| < 1$, i.e., when $R > 0$. The same result can also be derived by letting $H_{v_1}(z) = H_{v_2}(z) = 1, H_{i_1}(z) = R_1(z) = R$ and $H_{i_2}(z) = R_2(z)$ and choosing the structure of Fig. 2(b).

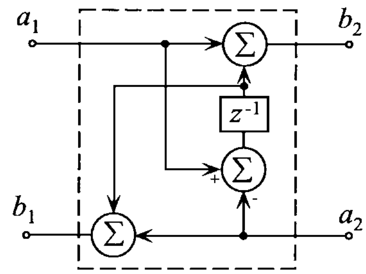

Fig. 3. Structure of the $R - C$ mutator. Notice that when port 2 is connected to a nonlinear capacitor, the wave variables a_2 and b_2 become b and a, respectively.

Notice that we may eliminate the instantaneous reflections at both ports by letting $R = T/2C$, in which case, we have $K(z) = z^{-1}$. This last result is quite interesting as the junction between a capacitive RTF and resistive one is performed by the scattering junction of Fig. 1(b), whose reflection coefficient is a delay element and whose second port is left unconnected. As a consequence, the whole scattering junction may be replaced with just one delay element, as predicted by the classical WDF theory [1] (see Section II-B).

The wave $R - C$ mutator can be used to extend the results on nonlinear resistors [11] to the case of nonlinear capacitors by following the method explained in Section III-A1. In fact, the waves at port 2 of the $R - C$ of the mutator are

$$\begin{aligned} A_2(z) &= V(z) + \frac{T}{2C}\,\frac{1+z^{-1}}{1-z^{-1}}\,I(z) = V(z) + \frac{1}{C}\,Q(z) \\ B_2(z) &= V(z) - \frac{T}{2C}\,\frac{1+z^{-1}}{1-z^{-1}}\,I(z) = V(z) - \frac{1}{C}\,Q(z) \end{aligned} \tag{13}$$

where $Q(z)$ is associated with the electrical charge $q(t), \dot{q}(t) = i(t)$. As a nonlinear capacitor can be described by an algebraic relationship of the form $P(v, q) = 0$ between the electrical charge q and the voltage v, we can use the results of Section II-C by letting C play the role of "reference capacitance" in the linear transformation that maps the Kirchhoff characteristic of the nonlinear capacitor onto the wave domain. In conclusion, in order to implement the nonlinear capacitor in the wave domain, we only need to implement a nonlinear map of the form $b = \tilde{f}(a)$ and connect it with the capacitive port of the $R - C$ mutator of Fig. 3, through the relationships $a = b_2$ and $a_2 = b$. This operation is possible if the invertibility conditions of the nonlinear characteristic of the capacitor are satisfied. Examples of application of this solution are reported in [15] for the anharmonic oscillator and in [16] for the inverted pendulum.

2) Nonlinear Inductors—$R - L$ Mutator: The $R - L$ mutator is a scattering junction between an inductive RTF and a resistive one; therefore, it can be implemented as in Fig. 1(b). With reference to the results of Section III-B, let $R_1(s) = R$ and $R_2(s) = sL, L > 0$. The corresponding scattering digital filter

$$K(z) = -\frac{p + z^{-1}}{1 + pz^{-1}}, \quad p = \frac{T - 2L/R}{T + 2L/R}. \tag{14}$$

is a first-order causal allpass filter whose stability is guaranteed by the condition $|p| < 1$, which requires the reference resistance R to be positive. Instantaneous reflections can be eliminated at both ports of the scattering junction by letting $R = 2L/T$, in which case, we have $K(z) = -z^{-1}$, as expected from the classical WDF theory [1].

The above $R-L$ wave mutator can be employed for implementing nonlinear reactive elements of the form $M(v,j) = 0, j(t) = di(t)/dt$. This definition does not correspond exactly to that of a nonlinear inductor, but it has a meaningful interpretation for mechanical systems. In fact, its characteristic represents a nonlinear relationship between force (voltage-like variable) and acceleration (derivative of the velocity, which is the current-like variable). In this case, we may proceed exactly as seen for nonlinear capacitors by defining the wave pair

$$A_2(z) = V(z) + \frac{2L}{T}\,\frac{1-z^{-1}}{1+z^{-1}}\,I(z) = V(z) + LJ(z)$$
$$B_2(z) = V(z) - \frac{2L}{T}\,\frac{1-z^{-1}}{1+z^{-1}}\,I(z) = V(z) - LJ(z)$$

where $J(z)$ is the Z-transform of the discretized version of $j(t)$, and L is a constant parameter that plays the role of a "reference inductance." The corresponding $R-L$ mutator results as being a dynamic scattering junction, whose reflection filter is just a delay element with sign change.

More interesting is the case in which the nonlinear inductor is defined as an algebraic relationship of the form $F(\phi, i) = 0$ between the flux linkage ϕ ($\dot{\phi} = v$) and the current i. Nonlinear inductors with a magnetic core are usually described this way. In this case, we can use the results of Section III-A2 with the pair of wave variables

$$A_2(z) = P(z) + LI(z) = \frac{T}{2}\,\frac{1+z^{-1}}{1-z^{-1}}\,V(z) + LI(z)$$
$$B_2(z) = P(z) - LI(z) = \frac{T}{2}\,\frac{1+z^{-1}}{1-z^{-1}}V(z) - LI(z)$$

which correspond to (8) when $H_{v_1}(z) = 1, H_{i_1}(z) = R, H_{i_2}(z) = L$ and

$$H_{v_2}(z) = \frac{T}{2}\,\frac{1+z^{-1}}{1-z^{-1}}.$$

From (9) we derive the reflection filter, which results as in (14), as expected.

As already seen in Section III-A2, we may choose between two alternative structures for the scattering junction ($R-L$ mutator), which are specified by (10) and (11). The structure of Fig. 2(b), however, is preferable to the one of Fig. 2(a), as $H_{i_1}(z)/H_{i_2}(z) = R/L = 2/T$ is just a scale factor, which can be easily embedded into the nonlinear characteristic of the inductor. The final implementation of the nonlinear inductor is thus represented by the $R-L$ mutator of Fig. 4, whose port 2 is closed on a nonlinear element, provided that the conditions of invertibility of the nonlinear characteristic of the inductor are satisfied.

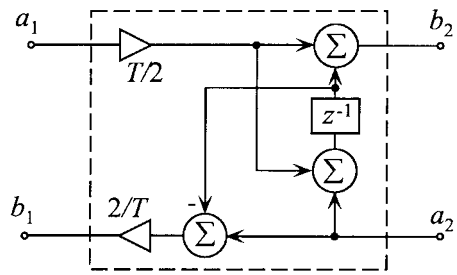

Fig. 4. Structure of the $R-L$ mutator. Notice that when port 2 is connected to a nonlinear inductor, the wave variables a_2 and b_2 become b and a, respectively. Notice also that the two multipliers are to be moved from the *resistive* port to the *inductive* port and then embedded into the nonlinearity.

D. Multiport Junctions with Memory

The approach proposed above for deriving scattering junctions with memory can be readily extended to parallel or series multiport junctions.

When only the current is filtered (of the Kirchhoff variables to which the waves are referred), the multiport junctions turn out to be structured like those derived by Fettweis [1], provided that reflection coefficients are replaced by reflection filters. For example, a series connection of n ports with RTF's $R_1(z)$ to $R_n(z)$ is characterized by the Kirchhoff equations $V_1(z) + \cdots + V_n(z) = 0$ and $I_1(z) = \cdots = I_n(z)$. The Z-transforms of the mth output wave $B_m(z), m = 1, \cdots, n$, can thus be written as a function of all input waves $A_k(z), k = 1, \cdots, n$ as $B_m(z) = A_m(z) - \Gamma_m(z)(A_1(z) + \cdots + A_n(z))$, where

$$\Gamma_m(z) = \frac{2R_m(z)}{R_1(z) + \cdots + R_n(z)}$$

are the reflection filters, which are assumed to be causal and stable. The multiport junction is thus characterized by n reflection filters $\Gamma_k(z), k = 1, \cdots, n$, which are bound to satisfy the constraint

$$\sum_{k=1}^{n} \Gamma_k(z) = 2. \tag{15}$$

Therefore, as in the linear case, the number of "independent" ports is $n-1$. The fact that the reflection filters are bound to satisfy (15) can be used to simplify the structure of the junction. For example, by letting $R_1(z) + \cdots + R_{n-1}(z) = R_n(z)$, we obtain $\Gamma_n(z) = 1$ with the result that $B_n(z) = 0$. By doing so, we make the nth port reflection free.

The above condition of *total adaptation* is a very strong one, whereas computability is, in fact, guaranteed by a condition of *instantaneous adaptation*. It is not difficult to verify that when the port RTF's are rational functions of the form

$$R_k(z) = \frac{C_k(z)}{D_k(z)} = \frac{c_k(0) + \sum_{m=1}^{N_k} c_k(m) z^{-m}}{1 + \sum_{m=1}^{M_k} d_k(m) z^{-m}}$$

then the instantaneous adaptation is guaranteed by the condition

$$c_n(0) = \sum_{m=1}^{n-1} c_m(0).$$

Similarly, we have the case of parallel multiport junctions with memory. In this case, the instantaneous adaptation condition is expressed as

$$c_n(0)^{-1} = \sum_{m=1}^{n-1} c_m(0)^{-1}.$$

Notice that when the bipole connected to the kth port has a transfer function that matches the port's RTF, the bipole reflection filter becomes zero, and the corresponding junction port is left unconnected. As there is no need to implement the reflection filter of an unconnected port, total adaptation can be used as an effective way of simplifying the implementation structure, and this fact will be made clearer in Section IV.

In general, we can always decide to use single-filter waves throughout the circuit, except where the wave characteristic of a nonlinear element is given in terms of waves of different nature. This choice allows us to maximize the compatibility of wave digital structures with the traditional WDF structures. For the sake of generality, however, it is instructive to show how to derive multiport junctions with memory corresponding to double-filter waves.

As seen in Section III-B, double-filter waves can be transformed into single-filter waves through a two-port element such as that in the left dashed box of Fig. 2(a). This fact suggests to us that we could implement a multiport junction for filtered waves by adding such devices at the ports of a voltage-wave junction. This solution, however, may not be very efficient because of the number of filters to be implemented.

Let us consider an n-port series junction for filtered waves. By applying the continuity constraint $V_1(z)+\cdots+V_n(z)=0$ and $I_1(z)=\cdots=I_n(z)$, we obtain

$$B_m(z) = A_m(z) - \Gamma_m(z)\left(\frac{A_1(z)}{H_{v_1}(z)} + \cdots + \frac{A_n(z)}{H_{v_n}(z)}\right)$$

where

$$\Gamma_m(z) = \frac{2H_{i_m}(z)}{\dfrac{H_{i_1}(z)}{H_{v_1}(z)} + \cdots + \dfrac{H_{i_n}(z)}{H_{v_n}(z)}}.$$

The condition of instantaneous adaptation at port n is thus given by

$$\frac{h_{i_m}(0)}{h_{v_m}(0)} = \sum_{k=1,k\neq m}^{n} \frac{h_{i_k}(0)}{h_{v_k}(0)}$$

where the filter's coefficients are obviously defined. Similarly, the condition of instantaneous adaptation in the case of the parallel multiport junction is given by

$$\frac{h_{v_m}(0)}{h_{i_m}(0)} = \sum_{k=1,k\neq m}^{n} \frac{h_{v_k}(0)}{h_{i_k}(0)}.$$

Notice that when all ports are referred to the same types of filtered waves, i.e., when $H_{v_1} = \cdots = H_{v_n} = H_v$, then the voltage filter can be eliminated from the junction

$$B_m(z) = A_m(z) - \Gamma_m(z)(A_1(z) + \cdots + A_n(z))$$

where

$$\Gamma_m(z) = \frac{2H_{i_m}(z)}{H_{i_1}(z) + \cdots + H_{i_n}(z)}.$$

E. Passivity of Multiport Junctions

An important problem that needs to be carefully addressed is that of the *passivity* of the multiport junctions with memory defined in the previous sections.

In classical wave digital filters, in order to characterize properties such as passivity, nonenergicicity, or losslessness, a *pseudo-power* [1], [22] function is defined. Through this function, it is not difficult to show that all adaptors (parallel, series, and lattice), ideal transformers, gyrators, and circulators are nonenergic; reactances (capacitors and inductors), unit elements, and QUARL's are pseudolossless; resistances are pseudopassive. The definition of pseudopower provided in [1] and [22], however, does not help us characterize the passivity of the wave mutators introduced before. In fact, the two-port scattering junctions with memory seen above cannot be easily represented as full-synchronic[1] wave digital two-ports [1], [22], as the delay elements used for implementing the reflection filter cannot be assigned a meaningful value of port resistance. In order to characterize the passivity of the wave mutators, it is thus necessary to introduce the concept of *total complex power* entering the junction.

Let us consider an n-port scattering junction with memory characterized by the single-filter digital waves $(a_k, b_k), k = 1,\cdots,n$ and the corresponding RTF's $R_k(z), k = 1,\cdots,n$. The total complex pseudopower entering the n ports of the junction is defined as

$$\begin{aligned}\mathcal{P} &= \sum_{k=1}^{n} I_k^{(*)} V_k \\ &= \tfrac{1}{4} \sum_{k=1}^{n} (A_k - B_k)^{(*)} G_k^{(*)} (A_k + B_k)\end{aligned}$$

where $G_k(z) = 1/R_k(z)$ and the asterisk between parentheses denotes paraconjugation, i.e., $G^{(*)}(z) = G^*(1/z^*)$ (which is the only analytical continuation of the complex conjugation on the unit circle of the Z-transform plane). Notice that the above definition is consistent with that of the *steady-state pseudopower* provided in [1] and [23] as well as that of the total complex power provided in [24]. The above expression of the pseudopower can be easily rewritten in matrix form as

$$\begin{aligned}\mathcal{P} &= \tfrac{1}{4}(\boldsymbol{A}-\boldsymbol{B})^{(*)}\boldsymbol{G}^{(*)}(\boldsymbol{A}+\boldsymbol{B}) \\ &= \tfrac{1}{4}\,\boldsymbol{A}^{(*)}\boldsymbol{M}\boldsymbol{A}\end{aligned}$$

[1] Most conventional WD filters are full-synchronic, i.e., all arithmetic operations can be performed simultaneously at every periodically recurring instants.

where, in the matrix case, the asterisk between parentheses denotes *transposed* paraconjugation, $\boldsymbol{A} = [A_1 \cdots A_n]^T$ and $\boldsymbol{B} = [B_1 \cdots B_n]^T$ are the wave vectors, which are related to each other through the scattering matrix $\boldsymbol{S}(z)$ as $\boldsymbol{B} = \boldsymbol{SA}, \boldsymbol{G}$ is a diagonal $n \times n$ matrix whose diagonal elements are $G_k, k = 1, \cdots, n,$ and

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{G}^{(*)} - \boldsymbol{S}^{(*)}\boldsymbol{G}^{(*)} + \boldsymbol{G}^{(*)}\boldsymbol{S} - \boldsymbol{S}^{(*)}\boldsymbol{G}^{(*)}\boldsymbol{S} \\ &= (\boldsymbol{I} - S^{(*)})\boldsymbol{G}^{(*)}(\boldsymbol{I} + \boldsymbol{S})\end{aligned}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. From the above expression of the complex pseudopower, it is not difficult to show that if the port admittance matrix $\boldsymbol{G}(z)$ converges on the unit circle, then both parallel and series multiport junctions are nonenergic. For example, the total complex pseudopower entering a parallel n-port junction, whose equations are

$$\begin{aligned}B_k &= 2V - A_k \\ V &= \tfrac{1}{2} \sum_{k=1}^{n} \Gamma_k A_k\end{aligned}$$

where $\Gamma_k = 2G_k/G$ and $G = \Sigma_{k=1}^n G_k$ can be easily proven to be identically zero. In fact, since $A_k + B_k = 2V, k = 1, \cdots, n,$ and $(A_k - B_k)G_k = 2I_k, k = 1, \cdots, n,$ we have

$$\mathcal{P} = \tfrac{1}{4} \sum_{k=1}^{n} (A_k - B_k)^{(*)} G_k^{(*)} (A_k + B_k) = V \sum_{k=1}^{n} I_k^{(*)} = 0$$

provided that the port RTF's are stable. The same conclusions can be drawn with series multiport junctions with memory as well as mutators.

In conclusion, a parallel or series multiport junction is nonenergic, like the memoryless junctions seen, for example, in [1], provided that the RTF's are stable. Parallel and series dynamic multiport junctions can thus be rightfully called *dynamic adaptors*.

Similar conclusions can be drawn for dynamic multiport junctions that correspond to double-filter digital waves. In fact, when using waves of the form $A = H_v V + H_i I$ and $B = H_v V - H_i I+$, we have $V = (A + B)/(2H_v)$ and $I = (A - B)/(2H_i)$. As a consequence, the definition of the total pseudopower entering the junction is the same as before, provided that

$$G_k = \frac{1}{H_{v_k} H_{i_k}^{(*)}}.$$

F. Final Remarks

The generalized wave digital structures resulting from the above definitions of digital waves resemble those of classical WDF's, especially when using single-filter waves. However, because of the newly added filters involved in multiport junctions and bipoles, a few considerations are in order.

First of all, we are now confronted with an increased freedom in the construction of the wave digital structures, as the choice of the waves that can be adopted for a circuit port is much wider than with classical WDF's. The choice of such waves, in fact, depends on the "degree of adaptation" between the RTF's of junction ports and bipoles, which decides how much of the "dynamics" of the system will be incorporated into the adaptors. This increased freedom, however, must be carefully dealt with as there are some constraints that need to be taken into account.

First of all, we need to make sure that the passivity properties of the individual elements of the reference analog circuit are preserved by their WD counterpart. This, however, is automatically guaranteed by an appropriate choice of the analog-to-digital mapping (e.g., bilinear transformation), provided that some precautions be taken in the numerization process. Furthermore, we need to guarantee that the stability properties of the whole analog reference circuit are preserved by its WD counterpart.

As far as this last point is concerned, we have already verified that parallel and series multiport junctions are intrinsically nonenergic, provided that the port RTF's are stable. A computable interconnection through nonenergic junction of elements having the same passivity properties as the reference ones will preserve the stability properties of the reference analog circuit. However, we need to make sure that the quantization of the filter coefficients will not affect the continuity constraints on the junctions.

Another fact that needs to be stressed is that the conditions of computability expressed in this section are only local because they only guarantee that a port is reflection-free, but they do not tell us whether the whole circuit will be, in fact, computable. In order to make sure that a nonlinearity can actually be connected to a port, not only do we need to make the port reflection-free, but we also need to make sure that no other delay-free directed loop via an outer feedback path exists. This problem could arise from the presence of a second nonlinearity in the circuit.

In general, a classical WDF implementation of a linear circuit gives us only one degree of freedom in the global choice of the reference resistances. This degree of freedom is exploited whenever a resistive nonlinearity is included in the circuit, as we need to adapt the port where the element is connected. Something similar happens when modeling filtered algebraic nonlinearities in the WD domain, as a minimal condition of instantaneous adaptation, must be satisfied at the port of insertion of the nonlinearity. As a consequence, when two or more nonlinearities are present in the circuit, we cannot guarantee that they can all be incorporated in the WD structure through the approach devised in Section III. A case in which this can, in fact, be done is when the nonlinear elements belong to portions of the circuit that are "instantaneously decoupled" from each other through a delay element. This situation is not at all infrequent in musical acoustics, where resonating or reverberating structures are often modeled by means of digital *waveguides* (networks of delay lines interconnected through WDF-like multiport junctions). Such multiport elements, which can be seen as the distributed-parameter counterpart of WDF's, have a "decoupling" effect on wave digital structures.

When no decoupling multipoles are present in the WD circuit, we need to identify a minimal portion of the circuit that contains all nonlinear elements and to proceed with its global discretization.

IV. Examples of Applications

In this section, we present some simple examples of scattering junctions and elements that cannot be modeled with classical WDF principles. We will consider just three cases:

- *Memristor*: We use double-filter waves, but the two-port junction that connects a memristor to a standard WDF port has no actual scattering. Its structure is, in fact, made only of the left dashed block of Fig. 2, whose purpose is just to make the waves dimensionally homogeneous with each other.
- *Frequency-Dependent Negative Resistor*: Its implementation requires one-filter waves.
- *Varactor Oscillator*: This is the simulation of a circuit with chaotic behavior in critical conditions.

A. Memristor

The memristor [19] is a bipole characterized by an algebraic relationship of the form $F(\phi, q) = 0$, where $\dot{\phi} = v$, and $\dot{q} = i$. Its name is a contraction of *memory resistor* because it behaves like a resistor whose resistance (conductance) depends on the complete past history of its current (voltage).

Although the memristor is realized only in the form of an active circuit, such a two-port circuit element is considered a to be as basic as resistors, capacitors, and inductors. The peculiar behavior of the memristor makes it particularly useful in applications to device modeling and signal processing [19].

A *physical* example of a memristor [12] characteristic is given by $q = G \ \sin\varphi$. This relationship implies that $i = (G \ \cos\varphi)v$, which confirms the fact that the conductance depends on the past history of the voltage.

Let us consider the problem of connecting the wave equivalent of such a nonlinear element with a standard WDF port, which is characterized by the wave variables $A_1 = H_{v_1}V + H_{i_1}I, B_1 = H_{v_1}V - H_{i_1}I$, where $H_{v_1} = 1$ and $H_{i_1} = R$.

A natural way of choosing the wave variables, in order for the memristor to be implemented as an instantaneous element, is

$$A_2 = H_{v_2}V + H_{i_2}I = H_{v_2}V + \mu G_2' I = P + \mu Q$$
$$B_2 = H_{v_2}V - H_{i_2}I = H_{v_2}V - \mu G_2' I = P - \mu Q$$

where $H_{v_2} = 1/s, G_2' = 1/s$ and $H_{i_2} = \mu/s$, which is to be mapped onto the domain of the Z-transform through bilinear transformation.

With reference to the results of Section III-B on the dynamic adaptors, it is not difficult to realize that the scattering filter is, in fact, simply a reflection coefficient of the form $K = (\mu - R)/(\mu + R)$. The condition of instantaneous adaptation $\mu = R$ leads to $K = 0$, which makes the junction totally reflection free.

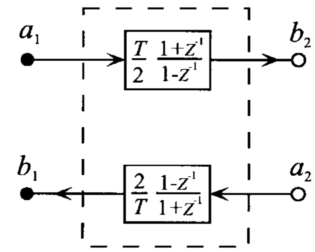

Fig. 5. Scattering junction for connecting the wave characteristic of the memristor to a standard WDF port. Notice that a_2 and b_2 will become the output and the input, respectively, of the instantaneous wave-equivalent of the memristor.

As far as the wave filters (i.e., those that appear in the left dashed box of Fig. 2) are concerned, we have

$$\frac{H_{i_1}}{H_{i_2}} = \frac{Rs}{\mu}\bigg|_{(2/T)(1-z^{-1}/1+z^{-1})} = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{H_{v_1}}{H_{v_2}}$$

as expected. In conclusion, the scattering junction that is required for connecting the wave equivalent of the memristor to a standard WDF port is as shown in Fig. 5.

Notice that the presence of poles on the unit circle could give rise to problems of stability, thus impairing the nonenergicity condition. In order to avoid this, we can adopt different discretization mappings from the bilinear one.

The condition of invertibility of the nonlinear characteristic $q = G \ \sin\varphi$ of the memristor can be readily derived with reference to the case of the voltage-controlled resistor. In fact, we have $G \ \cos\varphi \neq -1/R$. Global invertibility is guaranteed by the condition $G < 1/R$.

B. Frequency-Dependent Negative Resistor

A frequency-dependent negative resistor (FDNR) [12], [25] is defined by a relationship of the form $i = Md^2v/dt^2$. Its analog transfer function is, thus, of the form $V = (1/s^2M)I$, where s is the complex variable of the Laplace transform, which is to be remapped onto the Z-plane through bilinear transformation. Although the FDNR is a linear device, it cannot be implemented in the wave digital domain by using classical WDF adaptors. In fact, if we adopt the wave pair (a, b), which is referred to as the resistance R, we obtain a wave relationship of the form $B(z) = K(z)A(z)$, where

$$K(z) = \frac{\dfrac{1}{s^2M} - R}{\dfrac{1}{s^2M} + R}\Bigg|_{s=(2/T)(1-z^{-1}/1+z^{-1})}. \tag{16}$$

This reflection filter is not stable as its poles lie on the imaginary axis of the s plane. As the bilinear transformation preserves stability properties, the wave digital reflection filter will not be stable either. In fact, the filter we obtain, after choosing $M = T^2/(4R)$ for instantaneous adaptation, is of the form $K(z) = 2z^{-1}/(1 + z^{-2})$, whose poles are on the unit circle, as expected. Likewise, it is not possible to find a stable implementation of an $R - M$ mutator, as its reflection filter would, once again, be given by (16).

On the other hand, we can always model the FDNR with waves of the form $a = v + q/C$ and $b = v - q/C$, where

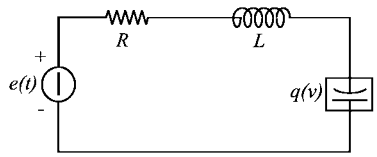

Fig. 6. Electrical circuit of the anharmonic oscillator.

$dq/dt = i$, and C plays the role of a "reference capacitance." With reference to such waves, the FNDR becomes a simple delay element, provided that the adaptation condition $M = CT/2$ is satisfied.

C. Varactor Oscillator

Examples of simulation of chaotic circuits in the wave digital domain are already available in the literature [11], [13]. In particular, Meerkötter [11] showed that as all chaotic circuits belonging to Chua's family [26] are characterized by the presence of a nonlinear resistance (which can be usually modeled with a piecewise linear characteristic), they can be easily implemented in the wave domain by means of the method of Section II-C.

Chaotic behavior in electrical circuits is due, in most cases, to a nonlinear resistance. There are, however, several examples of circuits that contain a nonlinear reactance and exhibit, in certain conditions, chaotic dynamics or particularly interesting phenomena such as subharmonic oscillation (period doubling). Examples of such circuits can be found in [27]–[31], and the accuracy of their computer simulation is usually quite sensitive to the errors caused by discretization.

An example of circuits of the type described above, whose simulation in the wave digital domain was studied in depth by Felderhoff [13], is represented by the anharmonic oscillator [29] of Fig. 6. This simple RLC circuit is characterized by a nonlinear voltage-controlled capacitance, whose $q - v$ characteristic

$$q = C_0 \frac{v}{\sqrt{1 + v/v_0}}, \quad v > -v_0$$

is shown in Fig. 7. The parameters used for the simulation of such a circuit are $v_0 = 0.6$ V, $R = 180\ \Omega, L = 100\ \mu$H, and $C_0 = 80$ pF, and the voltage supplied by the ideal generator is $e(t) = e_0 \sin(2\pi f_0 t), f_0 = (2\pi\sqrt{LC_0})^{-1}$.

When $v \leq v_0$, the nonlinear element is replaced by a resistive source $i = \dot{q} = (v + v_0)G, G > 0$. In any case, the chaotic behavior of the varactor occurs in the region $v > v_0$ of its characteristic.

In order to implement the varactor oscillator in the wave digital domain, Felderhoff [13] proposed a solution that employs classical WDF elements, including a transformer whose transform ratio $n = v'/v$ results as being a function of the nonlinear capacitor voltage v. As v depends on both port waves, we cannot obtain its value directly from the wave variables at the capacitor; otherwise, the implementation would be noncomputable. In order to overcome this problem, v should be derived by solving an implicit equation per sample with a consequent increment of complexity.

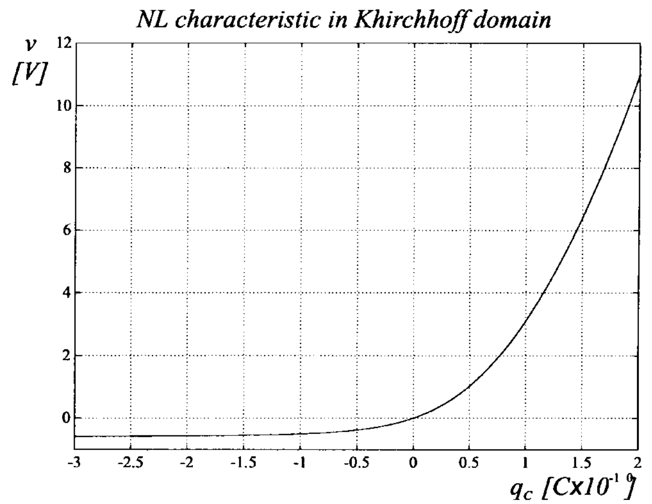

Fig. 7. Nonlinear characteristic of the capacitor of the anharmonic oscillator in the Kirchoof domain.

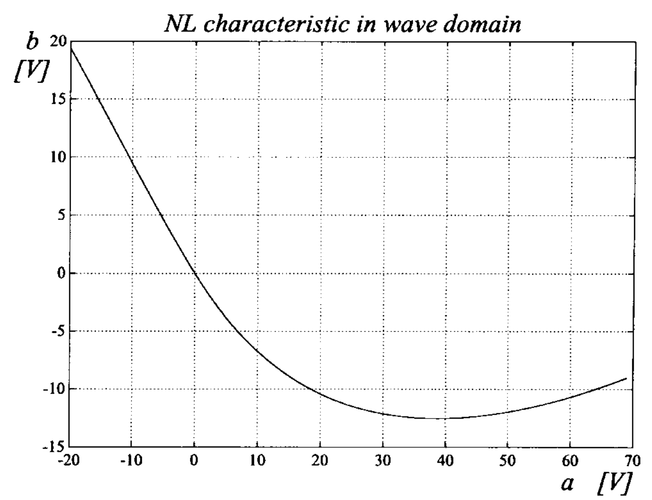

Fig. 8. Nonlinear characteristic of the capacitor of the anharmonic oscillator in the wave domain.

The varactor's circuit can be implemented in the wave domain by implementing the nonlinear capacitor as shown in Section III-C1. By adopting the waves (13), the characteristic of the nonlinear capacitor shown in Fig. 7 is mapped into the wave domain as shown in Fig. 8.

Now, we need to connect the nonlinear capacitor to the rest of the circuit. This operation may be done in two different ways, depending on whether we decide to apply the condition of instantaneous adaptation with the $R-L$ pair or that of total adaptation.

Performing instantaneous adaptation allows us to implement the linear portion of the circuit as a classical WDF structure. In Fig. 9, we can see the complete wave implementation of the varactor oscillator. Of the two scattering junctions of Fig. 9, one is a standard three-port series adapted junction. The first impedance port is set equal to $R_1 = R$ in order to include the resistor. The second port resistance is set equal to $R_2 = 2L/T$ in order to model the linear inductor as a simple delay with sign change. The third port is adapted $(R_3 = R_1 + R_2)$ so that no delay-free loops are created with the nonlinearity through

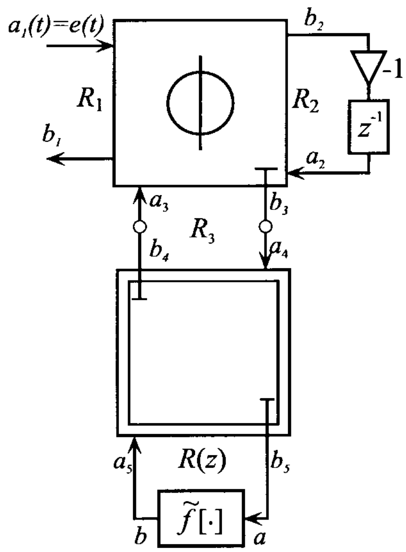

Fig. 9. Wave implementation of the anharmonic oscillator based on instantaneous adaptation. The double-bordered box represents an R-C mutator, and the presence of two "stubs" in its outputs denotes the absence of local instantaneous reflections.

the scattering junction with memory. The $C-R$ wave mutator is a scattering junction with memory where the reflection filter is $K = z^{-1}$, as seen in Section III-C1. In order for this to be true, we need to let

$$C_1 = \frac{T}{2R_3} = \frac{T/2}{R + 2L/T}.$$

This is the reference capacity that we need to use to determine the wave equivalent of the NL characteristic of the capacitor.

An alternative implementation can be obtained by performing total adaptation with respect to the whole $R-L$ portion of the circuit. This corresponds to defining a pair of wave variables that incorporate the whole memory of the linear part of the circuit. In other words, we connect the nonlinear capacitor to a voltage generator with internal impedance $Z_1 = R + sL$ (to be remapped onto the domain of the Z-transform through bilinear transformation) and perform adaptation with respect to the whole internal impedance. When the waves are referred to Z_1, the wave equivalent of the above generator is simply a source of voltage e. As a consequence, we only need a scattering junction with memory that changes the reference impedance $Z_2 = 1/(sC)$ to $Z_1 = R + sL$. Such a junction can be implemented as shown in Fig. 10, where the reflection filter is given by

$$K = \frac{z^{-1}(2-\delta) + z^{-2}\delta}{1 + z^{-1}\delta + z^{-2}(1-\delta)} \tag{17}$$

where $\delta = 2RC/T$, and

$$C = \frac{T/2}{R + 2L/T}$$

is the condition of instantaneous adaptation for the scattering filter.

The conditions under which the scattering filter is stable can be quite easily determined by studying the polynomial at the denominator of (17). Since this polynomial is Hurwitz for $0 < \delta \leq 1$, such conditions are simply $0 < RC \leq T/2$.

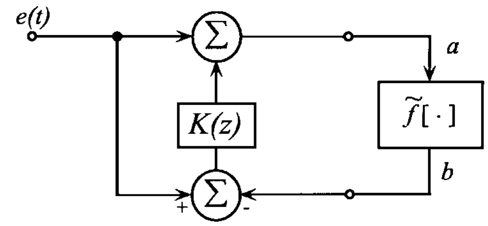

Fig. 10. Wave implementation of the anharmonic oscillator based on total adaptation. Notice that we do not need to draw a complete scattering cell with memory as we already know that the reflected wave toward the linear part of the circuit is zero (total adaptation).

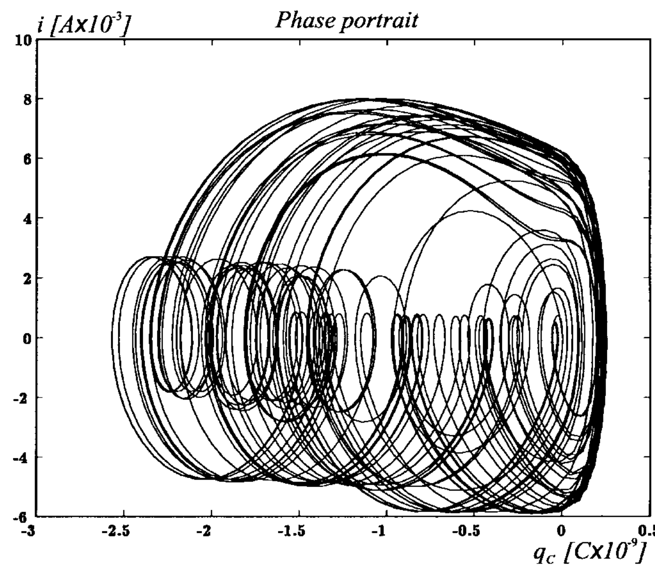

Fig. 11. Phase portrait of $\epsilon_0 = 3.57$ V using the parallel integrator and $T = 1/(32f_0)$.

A phase portrait of the varactor's state variables is shown in Fig. 11 for $T = 1/(32f_0)$. The accuracy of the simulation is quite independent of the sampling frequency, as long as the stability condition is satisfied, and the nonlinear element does not broaden the signal's bandwidth beyond the Nyquist frequency. In other situations [13], the simulation was quite sensitive to discretization problems, and the choice of the sampling frequency was critical.

V. CONCLUSIONS

In this paper, we proposed a generalization of the wave digital filter theory, whose aim is to enlarge the class of nonlinearities that can be embedded into WD structures. The class of nonlinear elements that can be modeled through the ideas proposed in this paper is that of the filtered algebraic nonlinearities, which covers a rather wide class of dynamic nonlinear elements. In particular, we introduced a class of dynamic multiport junctions that synergically combine together ideas of nonlinear circuit theory (mutators) and WDF theory (adaptors). We also showed that under some conditions on the reference port transfer functions, such junctions are nonenergic.

We showed that this generalization provides us with a certain freedom in the design of WD structures. In fact, not only can we design a dynamic adaptor in such a way to incorporate the whole dynamics of a nonlinear element into it, but we can also design a dynamic adaptor that will incorporate an arbitrarily large portion of a linear circuit.

The ideas presented in this work give us a different perspective on classical WDF's and, at the same time, provide us with a link to classical nonlinear circuit theory. In fact, they allow the designer to choose among a variety of alternative implementative solutions for each nonlinear circuit under examination, whereas all traditional WDF structures can be obtained as a particular case of the proposed approach. The enhanced flexibility in the design of the new WD structures is paid for in terms of conceptual complexity of the resulting structure, which complicates the automatic synthesis of WD systems.

Further extensions of the proposed theory are currently under study in order to include a wider class of nonlinear elements and circuits, including multiport nonlinearities with memory, which are a direct extension of nonlinear algebraic multiport devices [12].

REFERENCES

[1] A. Fettweis, "Wave digital filters: Theory and practice," *Proc. IEEE*, vol. 74, pp. 327–270, Feb. 1986.

[2] ——, "Some principles of designing digital filters imitating classical filter structures," *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 314–316, Mar. 1971.

[3] A. Fettweis and K. Meerkötter, "On adaptors for wave digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 516–525, Dec. 1975.

[4] A. Fettweis, "Digital circuits and systems," *IEEE Trans. Circuits Syst.*, vol. CAS-31, pp. 31–48, Jan. 1984.

[5] A. Fettweis, "Pseudopassivity, sensitivity, and stability of wave digital filters," *Trans. Circuit Theory*, vol. CT-19, pp. 668–673, Nov. 1972.

[6] A. Fettweis and K. Meerkötter, "Suppression of parasitic oscillations in wave digital filters," *Trans. Circuits Syst.*, vol. CAS-22, pp. 239–246, Mar. 1975.

[7] G. Borin, G. De Poli, and A. Sarti, "Sound synthesis by dynamic systems interaction," in *Readings in Computer Generated Music,* D. Baggi, Ed.. New York: IEEE Comput. Soc., 1992, pp. 139–160.

[8] ——,, "Algorithms and structures for synthesis using physical models," *Comput. Music J.*, vol. 16, pp. 30–42, 1992.

[9] ——, "Musical signal synthesis," in *Musical Signal Processing*, C. Roads, S. T. Pope, A. Piccialli, and G. De Poli, Eds. Milan, Italy: Swets and Zeitlinger, 1997, pp. 5–30.

[10] J. O. Smith, "Acoustic modeling using digital waveguides," in *Musical Signal Processing*, C. Roads, S. T. Pope, A. Piccialli, and G. De Poli, Eds.. Milan: Italy, Swets and Zeitlinger, 1997, pp. 221–263.

[11] K. Meerkötter and R. Scholtz, "Digital simulation of nonlinear circuits by wave digital filter principles," in *Proc. IEEE Intl. Symp. Circuits Syst.*, Portland, OR, May 8–11, 1989, vol. 1, pp. 720–723.

[12] L. O. Chua, "Nonlinear circuits," *IEEE Trans. Circuits Syst.*, vol. CAS-31, pp. 69–87, Jan. 1984.

[13] T. Felderhoff, "Simulation of nonlinear circuits with period doubling and chaotic behavior by wave digital principles," *Trans. Circuits Syst. I*, vol. 41, pp. 485–491, July 1994.

[14] S. A. Van Duyne, J. R. Pierce, and J. O. Smith, "Traveling-wave implementation of a lossless mode-coupling filter and the wave digital hammer," in *Proc. 1994 Int. Comput. Music Conf.*, Computer Music Assoc., 1994, pp. 411–418.

[15] A. Sarti and G. De Poli, "Generalized adaptors with memory for nonlinear wave digital structures," in *Proc. EUSIPCO, Eighth Euro Signal Process. Conf.*, Trieste, Italy, Sept. 10–13, 1996, vol. 3, pp. 1941–1944.

[16] T. Felderhoff, "A new wave description for nonlinear elements," *Int. Symp. Circuits Syst.*, Atlanta, GA, May 12–15, 1996, pp. 221–224.

[17] L. O. Chua, "Synthesis of new nonlinear network elements," *Proc. IEEE*, vol. 56, pp. 1325–1340, Aug. 1968.

[18] ——, *Introduction to Nonlinear Network Theory.* New York: McGraw-Hill, 1969.

[19] ——, "Memristor—The missing circuit element," *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 507–519, Sept. 1971.

[20] ——, "Device modeling via basic nonlinear circuit elements," *IEEE Trans. Circuits Syst.*, vol. CAS-27, pp. 1014–1044, Nov. 1980.

[21] ——, "Dynamic nonlinear networks: State-of-the-art," *IEEE Trans. Circuits Syst.*, vol. CAS-27, pp. 1059–1087, Nov. 1980.

[22] A. Fettweis, "Reciprocity, interreciprocity and transposition in wave digital filters," *Int. J. Circuit Theory Appl.*, vol. 1, pp. 323–337, Dec. 1973.

[23] ——, "Wave digital filters with reduced number of delays," *Int. J. Circuit Theory Appl.*, vol. 2, pp. 319–330, 1974.

[24] V. Belevitch, *Classical Network Theory*. San Francisco, CA; Holden Day, 1968.

[25] L. T. Bruton, *RC Active Circuits: Theory and Design*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[26] S. Wu, "Chua's circuit family," *Proc. IEEE*, vol. 75, pp. 1022–1032, Aug. 1987.

[27] M. Hasler and J. Neirynk, *Nonlinear Circuits*. Norwood, MA: Artech House, 1986.

[28] R. Madan, *Chua's Circuit: A Paradigm for Chaos*. Singapore: World Scientific, 1993.

[29] P. S. Linsay, "Period doubling and chaotic behavior in a driven anharmonic oscillator," *Phys. Rev. Let.*, vol. 47, pp. 1349–1352, 1981.

[30] J. Testa, J. Perez, and C. Jeffries, "Evidence for universal chaotic behavior of a driven nonlinear oscillator," *Phys. Rev. Let.*, vol. 48, pp. 714–717, 1982.

[31] C. Hayashi, *Nonlinear Oscillations in Physical Systems*. New York: McGraw-Hill, 1964.

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