Wave-based and Geometric Representations of Sound Fields

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Abstract

The two main approaches for the description of sound fields are methods derived from solutions of the wave equation and geometric methods based on analogies to ray optics. Their mathematical representations are reviewed and it is shown that representations by projective geometry and descriptions by Fourier acoustics lead to similar parametric representations of sound fields.

1 Introduction

Sound field synthesis comprises a number of methods for the creation of sound sensations by electro-acoustic means. The usual methods for home entertainment rely on stereophony with typically two to five loudspeakers. Recently also other sound reproduction methods with considerably more loudspeaker channels have been installed in public environments. The most popular of these recent methods are Ambisonics, wavefield synthesis, and vector-based amplitude panning. Typical applications are entertainment and artistic events. Consequently, the ultimate goal is the creation of audible spatial sensations for a large number of listeners.

Another application area is the reproduction of spatially distributed sound fields to create test environments for speech communication equipment. Rather than creating virtual realities for human listeners, the intention is here the physically correct reproduction of machine and street noise or the spatial reproduction of room reverberation.

The faithful reproduction of sound fields requires physically correct and mathematically tractable representations. There are two main approaches: one is based on the propagation of sound waves (wave-based methods) and the other on analogies to ray optics (geometric methods). Wave-based methods comprise the acoustic wave equation, integral relations involving Green's functions, and the expansion into circular and spherical harmonics [9]. Geometric methods include the mirror image source method, ray tracing, beam tracing, and certain extensions thereof.

Virtually all practical applications for the computation or reproduction of sound fields use either wave-based or geometric approaches. Since there is little cross-fertilization between both worlds, this contribution shows that connections can be established based on multidimensional signal theory. The description of acoustic signals in time and space and in the corresponding frequency domains [1, 3] provides new possibilities for geometric interpretations.

Sec. 2 gives a very concise account on the composition of sound fields from plane wave components. Sec. 3 presents a purely geometric interpretation of moving wave fronts in terms of projective geometry. An alternative route is given in Sec. 4 with a description of sound fields with Fourier transforms in time and space. Finally a unified view is developed in the concluding Sec. 5.

2 Representations of Sound Fields

This section presents a short account of a technique known as plane wave decomposition of a sound field [9]. The notion of a plane wave is the basis for the geometrical considerations in Sec. 3, while the signals introduced here are further analyzed by Fourier techniques in Sec. 4.

2.1 Overview

Three different representations of sound fields are presented here in increasing order of complexity. They are valid for typical physical quantities which describe a sound field, e.g. the sound pressure. The independent variables are the vector of space variables $\boldsymbol{x}$ and time t. A two-dimensional spatial extension with the scalar space variables x and y is used for simplicity, but the presented relations hold also for three spatial dimensions.

The following sound field representations are used here:

- a monofrequent plane wave with the frequency ω_0 and from the direction θ_0,
- a general plane wave from the direction θ_0, composed of monofrequent components,
- a general sound field, composed of plane waves from various directions.

These representations are described in more detail in the following sections.

2.2 Monofrequent Plane Wave

A complex-valued monofrequent plane wave is the extension of the idea of a phasor to a function of time and space

$$u(\boldsymbol{x},t;\theta_0,\omega_0) = U_0(\theta_0,\omega_0)\, e^{j(\boldsymbol{k}_0^{\mathrm{T}}\boldsymbol{x}+\omega_0 t)}\,. \tag{1}$$

It is called a plane wave because the dependency on space $\boldsymbol{x}$ and time t resembles a plane moving in space, as discussed in Sec. 3. The superscript T denotes transposition.

The so-called wave vector $\boldsymbol{k}_0$ is related to the angular frequency ω_0 and to the direction θ_0 from which the wave emanates by

$$\boldsymbol{k}_0 = \frac{\omega_0}{c}\,\boldsymbol{n}, \qquad \boldsymbol{n} = \begin{bmatrix} n_1(\theta_0) \\ n_2(\theta_0) \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} \tag{2}$$

The vector $\boldsymbol{n}$ is a unit vector determined by the direction of the wave θ_0. The complex amplitude $U_0(\theta_0,\omega_0)$ may vary with the angular frequency ω_0 and the direction θ_0.

The monofrequent plane wave (1) is a solution of the linear acoustic wave equation. Therefore all superpositions, including those presented in the following sections, are also representations of sound fields.

2.3 General Plane Wave

The superposition of monofrequent plane waves with different angular frequencies ω_0 gives a general plane wave propagating in a fixed direction

$$v(\boldsymbol{x},t;\theta_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\boldsymbol{x},t;\theta_0,\omega_0)\,\mathrm{d}\omega_0 = u_0\big(\theta_0, t+\frac{1}{c}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}\big)\,. \tag{3}$$

The wave form $u_0(\theta_0,t)$ is the inverse Fourier transform of the complex amplitude $U_0(\theta_0,\omega_0)$ with respect to ω_0

$$u_0(\theta_0,t) = \mathscr{F}_t^{-1}\{U_0(\theta_0,\omega_0)\} \tag{4}$$

with the Fourier transform $\mathscr{F}_t$ as defined in (18). The plane wave property from (1) is preserved, but the wave form $u_0(\theta_0,t)$ is arbitrary (as long as its Fourier transform exists).

2.4 General Sound Field

Finally a general sound field can be created by superposition of plane waves from all directions θ_0 as

$$w(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{0}^{2\pi} v(\boldsymbol{x},t;\theta_0)\,\mathrm{d}\theta_0\,. \tag{5}$$

The superposition of plane waves in (5) allows to create general sound fields, just like the superposition of monofrequent plane waves in (3) allows to generate plane waves with general wave form. Since plane waves are solutions of the acoustic wave equation, also their superposition (5) is a solution. This fact justifies to call $w(\boldsymbol{x},t)$ a sound field.

The plane wave property of $u(\boldsymbol{x},t;\theta_0,\omega_0)$ and $v(\boldsymbol{x},t;\theta_0)$ is the starting point for geometrical considerations in Sec. 3

3 Geometric Relations

The components $u(\boldsymbol{x},t;\theta_0,\omega_0)$ and $v(\boldsymbol{x},t;\theta_0)$ of the general sound field are called plane waves because they describe functions with constant values on a plane in space, or — in two spatial dimensions — on a line in a plane. This well-known fact is shown here with some simple arguments from projective geometry, a standard method in computer graphics and computer vision [2], but less so in acoustics. The geometrical description of the movement of a wave front requires shift operations as shown in Fig. 1. In Cartesian coordinates, such a shift cannot be calculated by a matrix multiplication. However, augmenting the coordinate vector by an additional element allows to compute mappings like translations and others by methods from linear algebra (see (10)).

3.1 Normal Form of a Line

The geometrical considerations start with the description of a line through the origin of the coordinate system (x,y), as shown in Fig. 1. When the position of the line is determined by the direction of a unit vector $\boldsymbol{n}$ as defined in (2), then the coodinates x and y of all points on the line satisfy the so-called normal form

$$\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x} = n_1 x + n_2 y = 0, \qquad \text{where} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6}$$

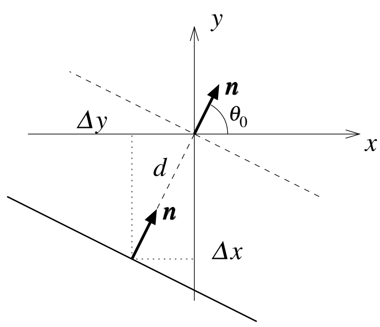

Figure 1: Two lines with normal vector $\boldsymbol{n}$ shifted by a distance d.

The point $\boldsymbol{x}$ can be mapped in the projective space introducing its homogeneous representation

$$\mathbf{x} = [x\ y\ 1]^{\mathrm{T}} = [\boldsymbol{x}\ 1]^{\mathrm{T}}\,. \tag{7}$$

The Cartesian coordinates are denoted by the two-element vector $\boldsymbol{x}$, while $\mathbf{x}$ is the corresponding vector of homogeneous coordinates.

As a consequence, the normal form in Eq. (6) can be rewritten by means of homogeneous coordinates

$$\mathbf{l}^T\mathbf{x} = 0\,, \tag{8}$$

where $\mathbf{l} = [n_1\ n_2\ 0]^T$ is the line parameter vector. Notice that all the vectors $\alpha\mathbf{l}$, with $\alpha \neq 0$, represent the same line: this means that the vector $\mathbf{l}$ is homogeneous as well.

The translated version of the line, is obtained by considering the translation vector $\boldsymbol{\Delta x} = [\Delta x\ \Delta y]^{\mathrm{T}}$, as shown in Fig. 1. In homogeneous coordinates the translated line is given by [2]

$$\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}\,, \tag{9}$$

with the translation matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

Eq. (9) can therefore be rewritten as

$$\mathbf{l}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\Delta x & -\Delta y & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ d \end{bmatrix} \tag{11}$$

with $-\Delta x n_1 - \Delta y n_2 = -\boldsymbol{n}^{\mathrm{T}}\boldsymbol{\Delta x} = d\,\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n} = d$. The elements $\boldsymbol{n}$ and d of the parameter vector are also called the representation of the line in the dual space or ray space [2].

The Hessian normal form in the coordinate system (x,y) follows by returning to Cartesian coordinates

$$\begin{bmatrix} n_1 \\ n_2 \\ d \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boldsymbol{n}^{\mathrm{T}}\boldsymbol{x} + d = 0\,. \tag{12}$$

3.2 A Travelling Wave Front

The shifted line is now interpreted as a plane wave front which moves with time at a constant speed c. Thus the distance d from the origin in (12) becomes a linear function of time $d = d(t) = ct$ and the line travels with constant speed in the direction of $-\boldsymbol{n}$ as shown in Fig. 2.

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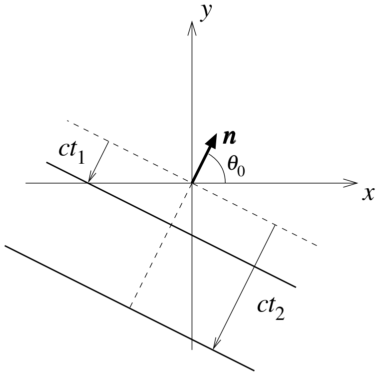

Figure 2: A travelling wave front for two different points in time t_1 and t_2.

The introduction of a time dependence is reflected by a new form of the Hessian normal form (12) as

$$\begin{bmatrix} \boldsymbol{n} \\ ct \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \boldsymbol{n}^{\mathrm{T}}\boldsymbol{x} + ct = 0. \tag{13}$$

The left side of (13) is the analytical description of a travelling wave front in homogeneous coordinates, the right side is the corresponding Hessian normal form.

3.3 Correspondence to the Plane Wave

The parameter representation in (13) is only unique up to a constant factor. To show the correspondence with the plane wave solutions introduced in Sec. 2, a special form of this factor is chosen.

In accordance with (2), the normal form (13) is multiplied by

$$k_0 = \frac{\omega_0}{c}, \qquad k_0\boldsymbol{n} = \boldsymbol{k}_0 \tag{14}$$

to obtain

$$\begin{bmatrix} \boldsymbol{k}_0 \\ \omega_0 t \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \boldsymbol{k}_0^{\mathrm{T}}\boldsymbol{x} + \omega_0 t = 0\,. \tag{15}$$

As a parameter description of a travelling line, the constants $\boldsymbol{k}_0$ and ω_0 have no physical meaning. However, as description of a plane wave, the right hand side of (15) corresponds to the argument of the exponential term in (1) where $\boldsymbol{k}_0$ and ω_0 represent the wave vector and the angular frequency, respectively.

The condition (14) can be expanded in homogeneous coordinates into the parameter representation of a circle

$$\begin{bmatrix} k_{01} \\ k_{02} \\ \omega_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/c^2 \end{bmatrix} \begin{bmatrix} k_{01} \\ k_{02} \\ \omega_0 \end{bmatrix} = 0. \tag{16}$$

The parametric description of a circle becomes obvious when converting to Cartesian coordinates through division by ω_0

$$\begin{bmatrix} k_{01} \\ k_{02} \\ \omega_0 \end{bmatrix} \rightarrow \begin{bmatrix} k_{01}/\omega_0 \\ k_{02}/\omega_0 \end{bmatrix} = 1/c \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = 1/c \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}\,. \tag{17}$$

This geometric description will be reviewed in the following section from the view point of Fourier acoustics.

4 Fourier Acoustics

This section discusses the sound field components from Sec. 2 from the view point of Fourier methods [3, 4, 9]. Here, the wave nature of these components is emphasized by application of Fourier transformations with respect to time and space.

4.1 Fourier Transforms for Time and Space

For a scalar function $u(\boldsymbol{x},t)$ of space $\boldsymbol{x}$ and time t, the Fourier-Transform with respect to time is defined by

$$U(\boldsymbol{x},\omega) = \int_{-\infty}^{\infty} u(\boldsymbol{x},t)e^{-j\omega t}\,\mathrm{d}t = \mathscr{F}_t\{\,u(\boldsymbol{x},t)\,\}\,, \tag{18}$$

and further the two-dimensional Fourier-Transform with respect to space

$$\bar{U}(\boldsymbol{k},\omega) = \iint_{-\infty}^{\infty} U(\boldsymbol{x},\omega)e^{-j\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} = \mathscr{F}_{\boldsymbol{x}}\{\,U(\boldsymbol{x},\omega)\,\}\,. \tag{19}$$

The Fourier transforms of exponential functions are given by Delta-functions in the corresponding frequency variables, the angular frequency ω and the wave vector $\boldsymbol{k}$

$$\mathscr{F}_t\{\,e^{j\omega_0 t}\,\} = 2\pi\,\delta(\omega-\omega_0) \tag{20}$$

$$\mathscr{F}_{\boldsymbol{x}}\{\,e^{j\boldsymbol{k}_0^{\mathrm{T}}\boldsymbol{x}}\,\} = (2\pi)^2\,\delta(\boldsymbol{k}-\boldsymbol{k}_0). \tag{21}$$

4.2 Fourier Transforms of Plane Waves

The relations from Sec. 4.1 are now applied to the plane waves introduced in Sec. 2.2 and 2.3. The Fourier transform of the monofrequent plane wave from (1) is given by

$$\begin{aligned} U(\boldsymbol{x},\omega;\omega_0,\theta_0) &= \mathscr{F}_t\{\,u(\boldsymbol{x},t;\omega_0,\theta_0)\,\} \\ &= U_0(\omega_0,\theta_0)\,e^{j\boldsymbol{k}_0^{\mathrm{T}}\boldsymbol{x}}\cdot 2\pi\,\delta(\omega-\omega_0) \end{aligned} \tag{22}$$

and subsequent spatial Fourier transform results in

$$\begin{aligned} \bar{U}(\boldsymbol{k},\omega;\omega_0,\theta_0) &= \mathscr{F}_x\{\,u(\boldsymbol{x},\omega;\omega_0,\theta_0)\,\} \\ &= U_0(\omega_0,\theta_0)\,(2\pi)^3\,\delta(\omega-\omega_0)\delta(\boldsymbol{k}-\boldsymbol{k}_0)\,. \end{aligned} \tag{23}$$

Thus the monofrequent plane wave (1) is represented by a single point $(\omega_0, \boldsymbol{k}_0)$ in the 3D parameter space of the angular frequency ω and the wave number $\boldsymbol{k} = [k_1\ k_2]^{\mathrm{T}}$ as shown on the left of Fig. 3.

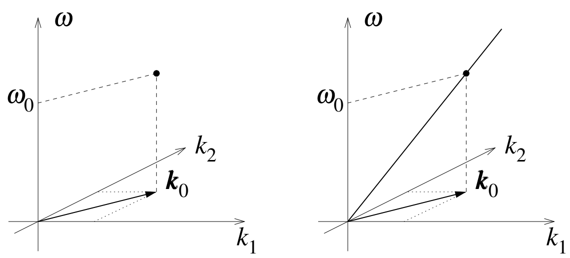

Figure 3: Space-time Fourier transforms of a monofrequent plane wave (left) and of a general plane wave (right).

Similar to Sec. 2.3, the frequency domain representation of a general plane wave follows from $\bar{U}(\boldsymbol{k},\omega;\omega_0,\theta_0)$

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by integration with respect to all angular frequencies ω_0 as

$$\bar{V}(\boldsymbol{k},\omega;\omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{U}(\boldsymbol{k},\omega;\omega_0,\theta_0)\,d\omega_0 \qquad (24)$$
$$= U_0(\omega,\theta_0)\,(2\pi)^2\cdot\delta(\boldsymbol{k}-\boldsymbol{k}_0)$$

Thus a general plane wave is represented in the 3D parameter space $(\boldsymbol{k},\omega_0)$ by a line though the origin, see Fig. 3, right.

Finally, the frequency domain description of a general sound field results from integration with respect to the angle θ_0, compare Sec. 2.4,

$$\bar{W}(\boldsymbol{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{V}(\boldsymbol{k},\omega;\omega_0)\,d\theta_0 \qquad (25)$$
$$= 2\pi\int_{0}^{2\pi} U_0(\omega,\theta_0)\cdot\delta\left(\boldsymbol{k}-\frac{\omega}{c}\boldsymbol{n}(\theta_0)\right)d\theta_0 .$$

Now the line from Fig. 3, right, rotates with varying angle θ_0, forming a cone as shown in Fig. 4. For each fixed value of ω, a horizontal cross section through the cone describes a circle. Its radius increases with increasing angular frequency, thus linking angular frequency and wave number. For acoustic waves, this strong coupling of temporal and spatial variations is given by (2), also called the dispersion relation.

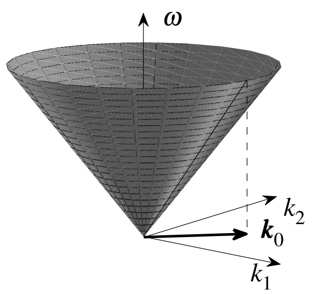

Figure 4: Space-time Fourier-Transform of a general sound field.

5 Conclusions

The representations (15) and (23) show that such diverse approaches as projective geometry and Fourier acoustics lead to rather similar parameter representations. The travelling line from Fig. 2 can to be interpreted according to (1) as a line of constant phase

$$\boldsymbol{k}_0^{\mathrm{T}}\boldsymbol{x}+\omega_0 t = \text{const}\,. \qquad (26)$$

The three-dimensional parameter space of projective geometry consists of the parameter vectors $(\boldsymbol{n}_0,d)$ or $(\mathbf{k}_0,\omega_0 t)$ in (15). The third dimension is generated as a homogeneous coordinate without any physical origin. In the geometrical context, c, ω_0 and t are parameters which discribe translations of the line from Fig. 2.

In Fourier acoustics, the wave front defined by (26) has a more general meaning. It is a line (or plane in 3D) of constant phase of a wave from (1) or (3) with the wave vector $\boldsymbol{k}_0$ and an angular frequency ω_0. Thus the axes in the corresponding three-dimensional frequency domain (see Fig. 3) are endowed with a clear physical identity.

It has been shown that projective geometry represents an effective tool for representing travelling waves. However, projective geometry itself may not be sufficient for describing more complex and realistic acoustic scenarios. For example, a desirable feature would be the ability to describe wall reflections, distinguishing between the two sides of a reflecting surface. This can be addressed employing the oriented projective geometry [7].

It introduces the concept of signed homogeneous coordinates, that allows to distinguish between a positive and a negative side of the projective space. This implies a number of advantages, for example the possibility to define oriented lines, oriented points, segments and directions.

In the literature, the oriented projective geometry has been mostly applied in Computer Vision (see for example [5, 8]). Its application in acoustics is quite novel: in [6] the authors show the derivation of a set of oriented geometric primitives suitable for the computation of the mutual visibility among acoustic reflectors. This paves the way for an efficient beam tracing algorithm. The key point is that oriented projective geometry allows to describe the two sides of an acoustic reflector in terms of the directions of rays passing through it.

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